QUANTUM ENGINEERING TOULOUSE
LCAR Laboratoire Collisions Agrégats Réactivité
Université de Toulouse
UNIVERSITÉ TOULOUSE III PAUL SABATIER
cnrs
in collaboration:
icb Laboratoire Interdisciplinaire Carnot de Bourgogne

Experiments with Bose-Einstein condensates

From atom cooling to quantum control and simulation

QuanTEEM Winter School
Dijon, 24/02/2025

Bruno Peaudecerf (CNRS)

Lecture II

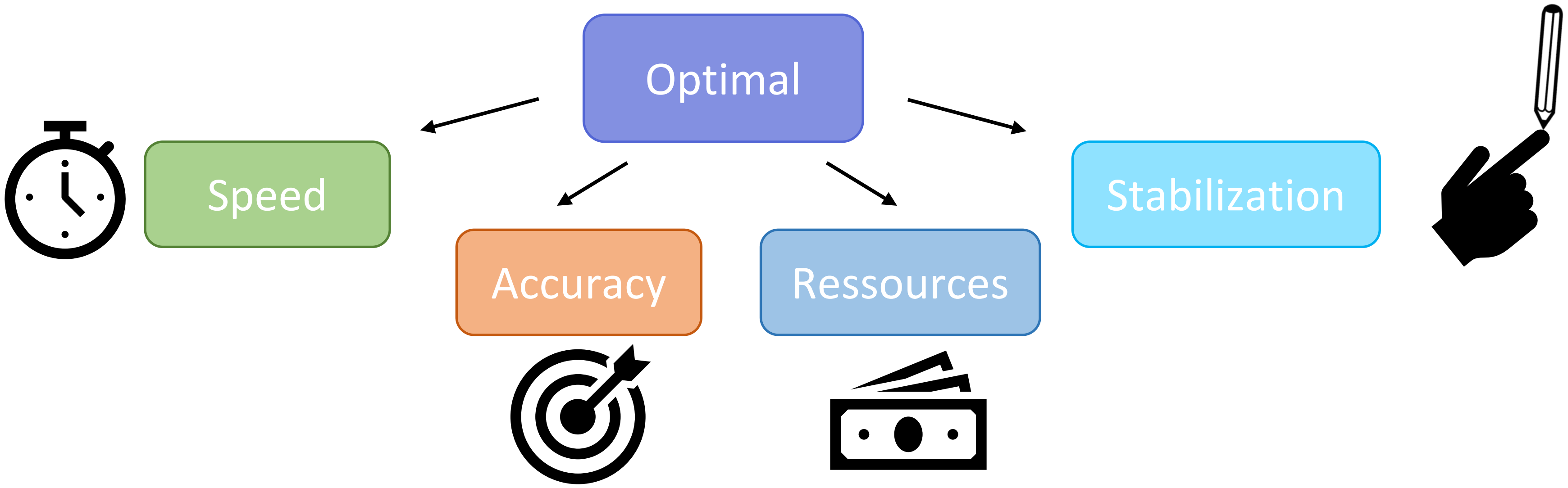
Quantum control with BEC

Introduction

Control theory: get a dynamical system to operate **optimally** within physical bounds

$$\dot{\mathbf{x}}(t) = \boldsymbol{f}\,(\mathbf{x}(t), \boldsymbol{u}(t), t)$$

Control parameter(s) $\boldsymbol{u}(\boldsymbol{t})$ **:** design interactions, energy landscapes,… to achieve goal



Introduction

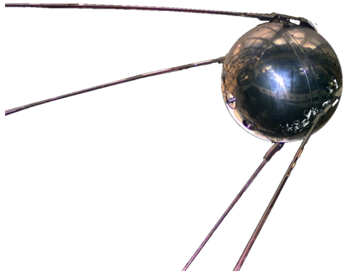
From Bernoulli's **brachistochrone** (1696)

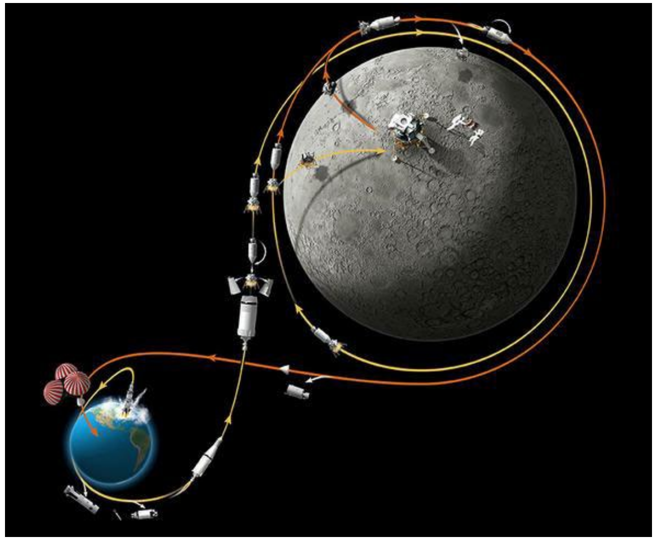
Museo Galileo
Florence

to Euler-Lagrange **multipliers** (1766)
for optimisation under constraints

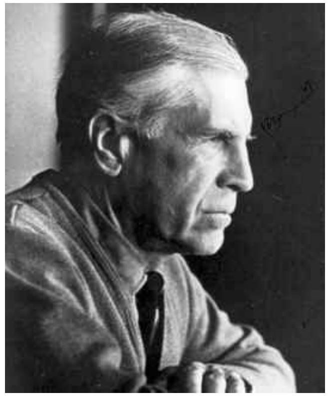
To optimal trajectories for spacecrafts (1960's)

Sputnik, Apollo…





Introduction



L. S. Pontryagin (1908-1988)

In the 1960's **optimal control theory** gets formalized:
a set of mathematical tools to find optimal solutions

Pontryagin Maximum Principle : a necessary condition

Quantum optimal control: extension to quantum systems

- Nuclear Magnetic Resonance
- Physical Chemistry
- *Quantum technologies (various platforms) :*
  NV centers, photonic states in cavity, atoms (Rydberg, quantum gases)…

Alongside **other control strategies** :
- Feedback theory
- Shortcuts to adiabaticity
- Machine learning, …

Introduction

Ultracold quantum gases are the result of increasing ***control*** of the atomic state

- Optical pumping (Kastler – Nobel 1966): *light can control populations in magnetic sublevels of the electron*

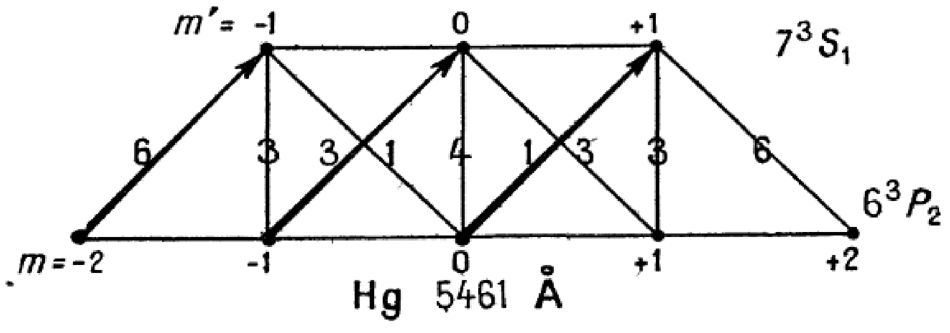
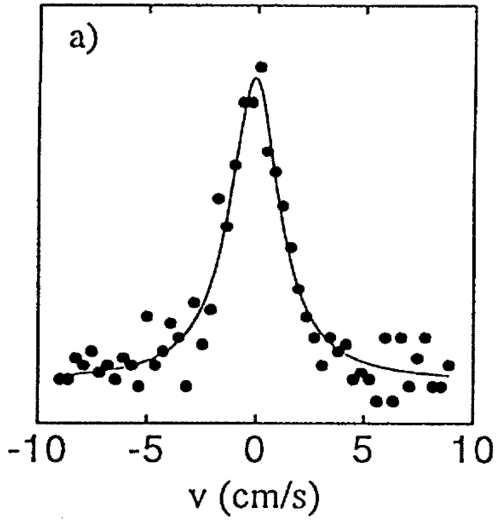


Fig. 2.

Kastler, *Jour. Phys. Radium* ***11****, 255* (1950)

- Laser cooling (Chu, Cohen-Tannoudji, Phillips – Nobel 1997): *optical pumping can affect external degrees of freedom of the atoms : cooling below the photon recoil limit*



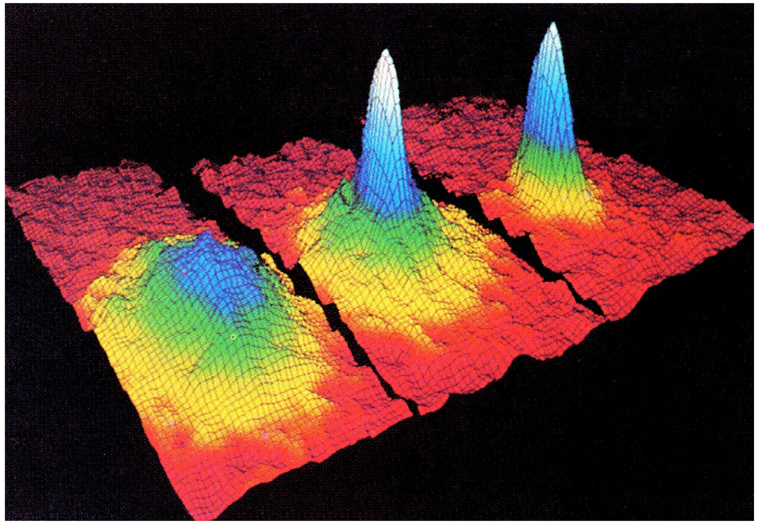
Lawall et al, *Phys Rev Lett* ***75****, 4194* (1995)

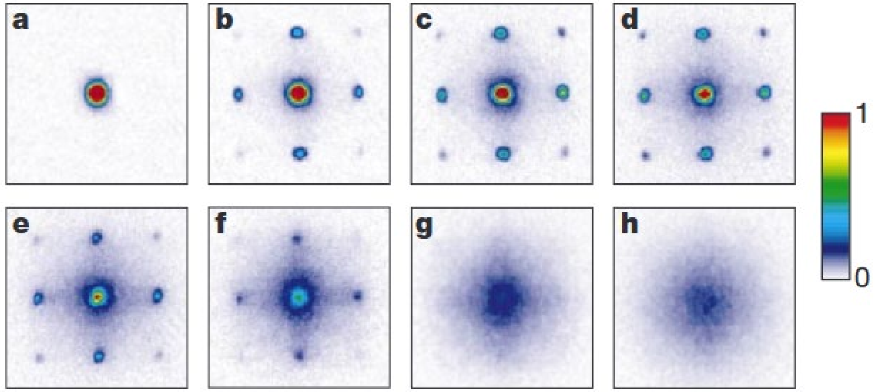
Introduction

Ultracold quantum gases are the result of increasing ***control*** of the atomic state

- Bose-Einstein condensation
  from evaporative cooling
  (Cornell, Ketterle, Wieman – Nobel 2001)
  atoms in a single quantum state

Anderson et al,
Science ***269**, 198* (1995)





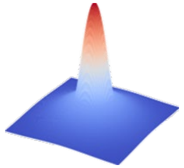
- Many-body quantum phase transitions :
  control of collective state of interacting atoms

Greiner et al,
Nature ***415**, 39 (2002)*

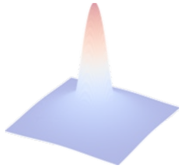
Outline

- BEC in a sine potential
- Optimal control in a sine potential
- State reconstruction
- Control with the non linearity

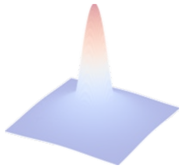
Outline



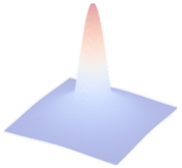
BEC in a sine potential



Optimal control in a sine potential

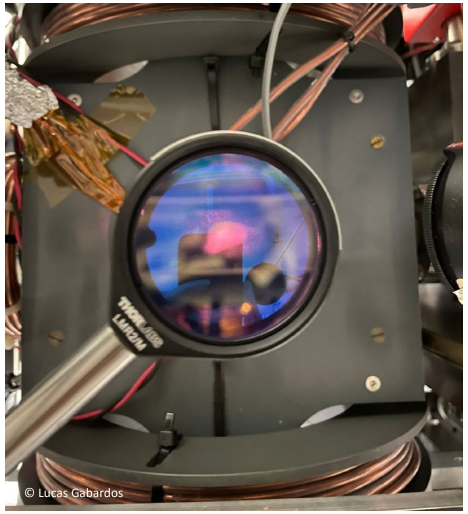


State reconstruction



Control with the non linearity

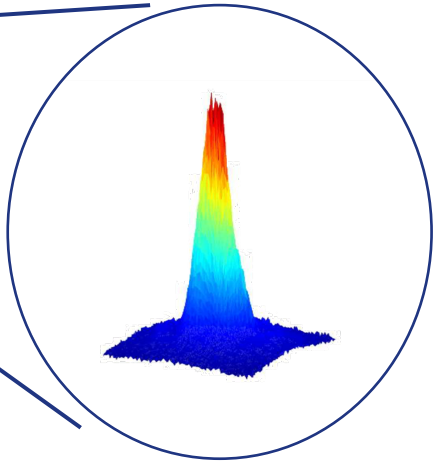
A cold atoms experiment (Toulouse)



Magneto-optical trap

A ^{87}Rb gas in a vacuum cell is :
- laser cooled
- magnetically trapped and evaporated
- trapped in far-off-resonant light beams (dipole trap),
- evaporated further… **untill condensation**

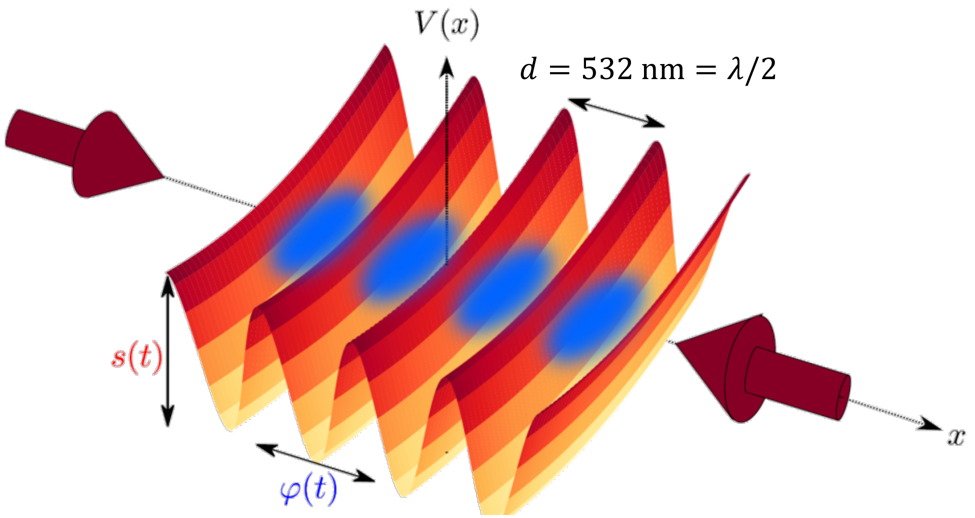
General view



Bose-Einstein condensates (BEC) of ^{87}Rb:
a **macroscopic matterwave**
($5 \cdot 10^5$ atoms at $T \simeq 90$ nK)

A matterwave in a sine potential

▪ BEC in an optical lattice potential



Laser beams **far detuned** (1064nm) from atomic transition (780nm)

Induces an **electric dipole** interacting with the field: **dipole force**
deriving from a **conservative dipole potential**

$$V_{dip} \propto \frac{I}{\Delta}$$

Light intensity

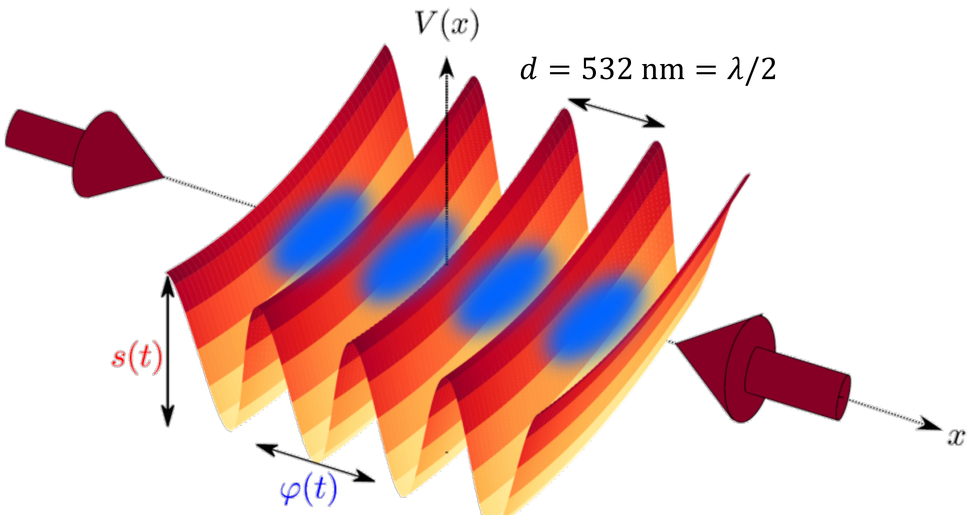
Detuning
(atomic transition-laser)

We can then create a perfect sine potential with retro-reflected laser beams

$$V \propto I \propto 4I_0 \sin^2(kx)$$

A matterwave in a sine potential

- ## BEC in an optical lattice potential



$$V(x,t) = \frac{s(t)E_L}{2}[1 - \cos(k_L x + \varphi(t))]$$

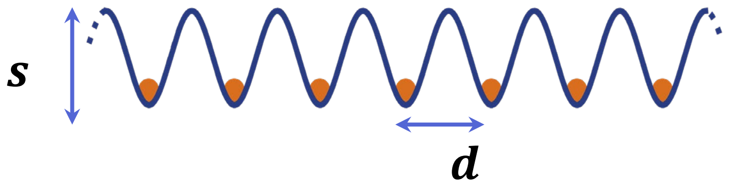
Characteristic quantities :

$$k_L = \frac{2\pi}{d},\ E_L = \frac{\hbar^2 k_L^2}{2m}$$

- Beams independently controlled with Acousto-Optic Modulators (AOM), changing phase and amplitude

→ We can vary the depth and position of the lattice **arbitrarily,** to **manipulate the BEC wavefunction**

Band structure



$$V(x,t) = \frac{s(t)E_L}{2}[1 - \cos(k_L x + \varphi(t))]$$

The lattice Hamiltonian is invariant by translation of d: $\hat{T}_d = \exp\left(-\frac{i\hat{p}d}{\hbar}\right)$

$$[\hat{T}_d, \hat{H}] = 0$$

Common eigenstates can be found.

Bloch's theorem:

The eigenstates of a periodic potential **are of the form:**

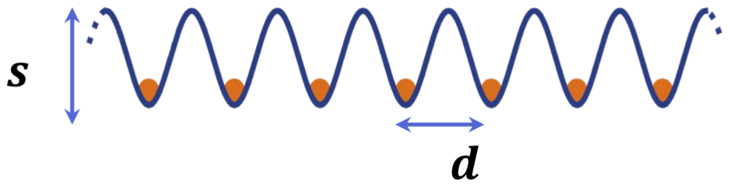
$$\Psi_{n,q}(x) = e^{\frac{iqx}{\hbar}} u_{n,q}(x)$$

with $u_{n,q}(x)$ a d-periodic function $u_{n,q}(x+d) = u_{n,q}(x)$

q denotes the class of eigenstates of $\hat{T}_d$:

$$\hat{T}_d \Psi_{n,q}(x) = e^{\frac{iqd}{\hbar}} \Psi_{n,q}(x)$$

Band structure



$$V(x,t) = \frac{s(t)E_L}{2}[1 - \cos(k_L x + \varphi(t))]$$

Eigenstates $\Psi_{n,q}(x) = e^{\frac{iqx}{\hbar}} u_{n,q}(x)$

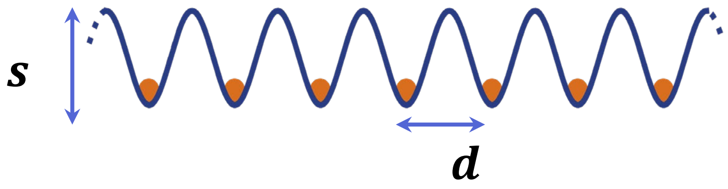
$u_{n,q}(x)$ periodic → Fourier series on plane waves

$$\Psi_{n,q}(x) = \sum_{\ell} c_\ell \, e^{\frac{iqx}{\hbar}} \frac{e^{i\frac{\ell k_L x}{\hbar}}}{\sqrt{d}} \qquad |\Psi_{n,q}\rangle = \sum_{\ell} c_\ell |\chi_{\ell k_L + q}\rangle, \qquad \hat{p}|\chi_p\rangle = p|\chi_p\rangle$$

Inject into stationary Schrödinger equation ($\varphi = 0$ for now):

$$\left(\frac{\hat{p}^2}{2m} + \frac{sE_L}{2}\cos(k_L \hat{x})\right)|\Psi_{n,q}\rangle = E_n(q)|\Psi_{n,q}\rangle$$

Band structure



$$V(x,t) = \frac{s(t)E_L}{2}\left[1 - \cos(k_L x + \varphi(t))\right]$$

Injecting into stationary Schrödinger equation,
gives the **central equation** (coupled equations on c_ℓ):

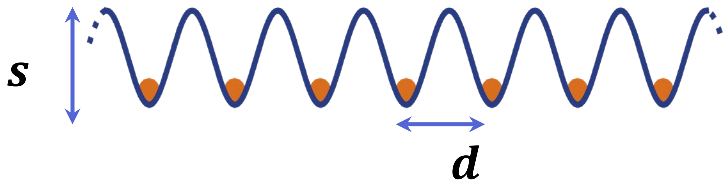
$$\left((\ell + q/k_L)^2 + \frac{s}{2}\right)c_{n,\ell}(q) - \frac{s}{4}\left(c_{n,\ell-1}(q) + c_{n,\ell+1}(q)\right) = \frac{E_n(q)}{E_L}c_{n,\ell}(q)$$

Matrix form:

$$C = \begin{pmatrix} \cdots \\ c_{\ell-1} \\ c_\ell \\ c_{\ell+1} \\ \cdots \end{pmatrix}, \qquad MC_n(q) = \frac{E_n(q)}{E_L}C_n(q), \qquad M = \begin{pmatrix} \ddots & -\frac{s}{4} & 0 \\ -\frac{s}{4} & (\ell + q/k_L)^2 + \frac{s}{2} & -\frac{s}{4} \\ 0 & -\frac{s}{4} & \ddots \end{pmatrix}$$

Solve with finite Hilbert space size N

Band structure

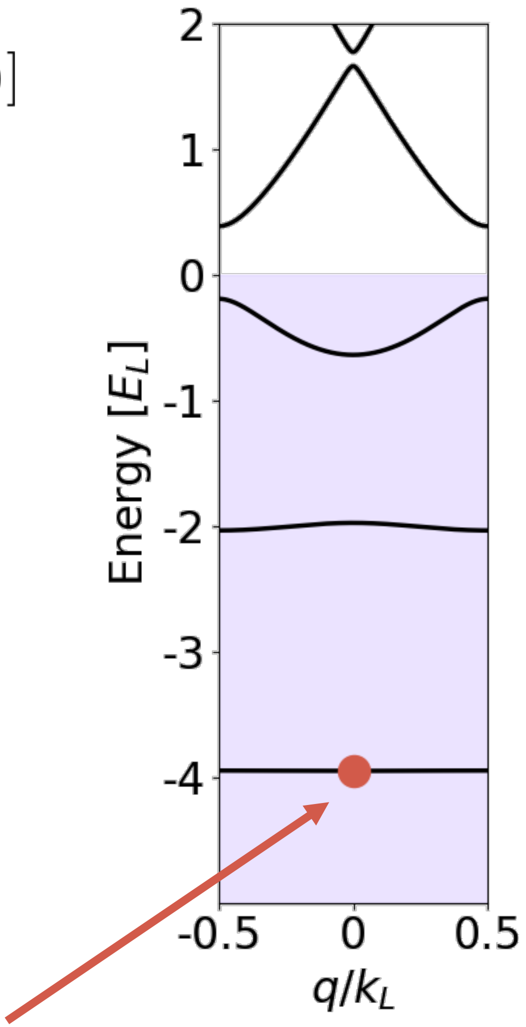


$$V(x,t) = \frac{s(t)E_L}{2}[1 - \cos(k_L x + \varphi(t))]$$

$$\left((\ell + q/k_L)^2 + \frac{s}{2}\right) c_{n,\ell}(q) - \frac{s}{4}\left(c_{n,\ell-1}(q) + c_{n,\ell+1}(q)\right) = \frac{E_n(q)}{E_L} c_{n,\ell}(q)$$

***Band structure* of the lattice levels $E_n(q)$**

Eigenstates are characterized by their coefficients $c_\ell^{(q,n)}$

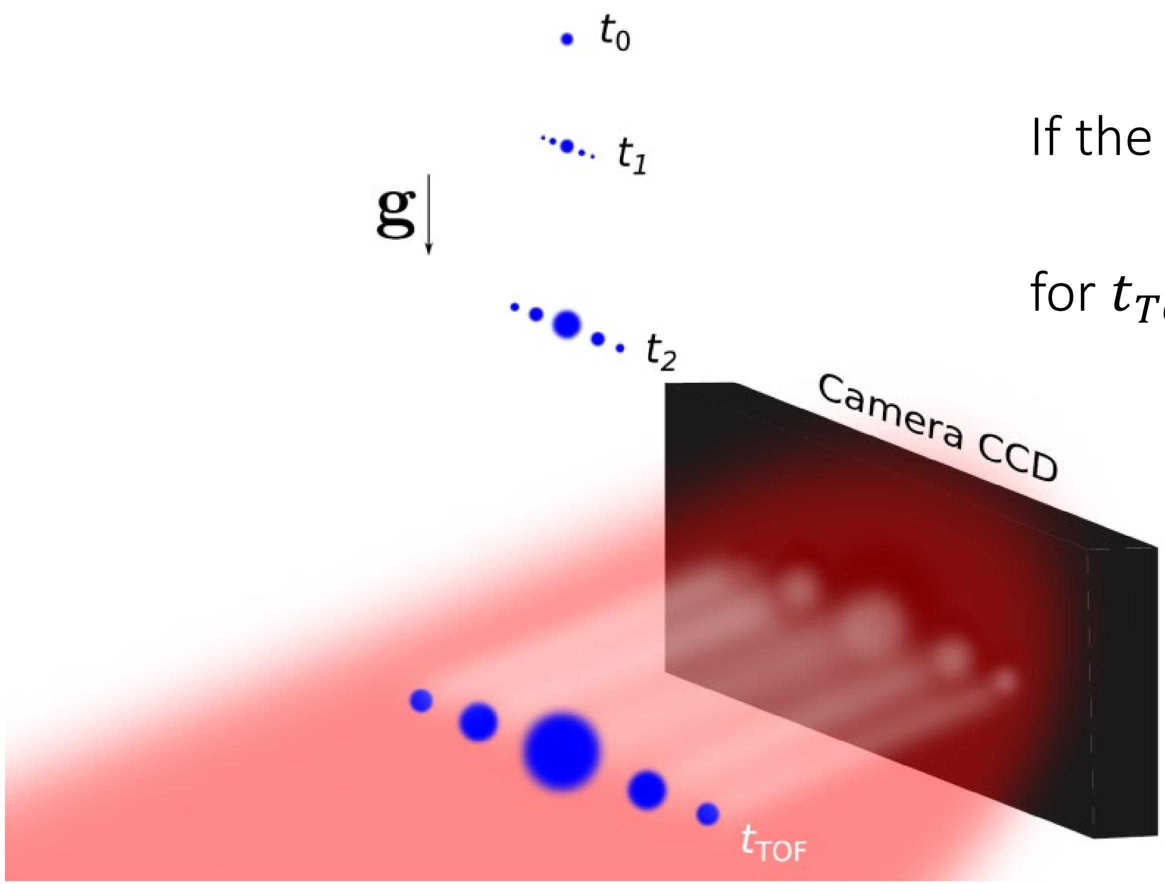


$(s = 5)$

In most experiments, we apply *adiabatically* the lattice potential on the BEC at rest ($p = 0$): we **prepare the lattice ground state**

What can we measure?

- We take an **absorption image** after a long time-of-flight



What happens:

$$x(t_{\mathrm{TOF}}) = x(0) + v(0)t_{\mathrm{TOF}}$$
$$\simeq v(0)t_{\mathrm{TOF}}$$

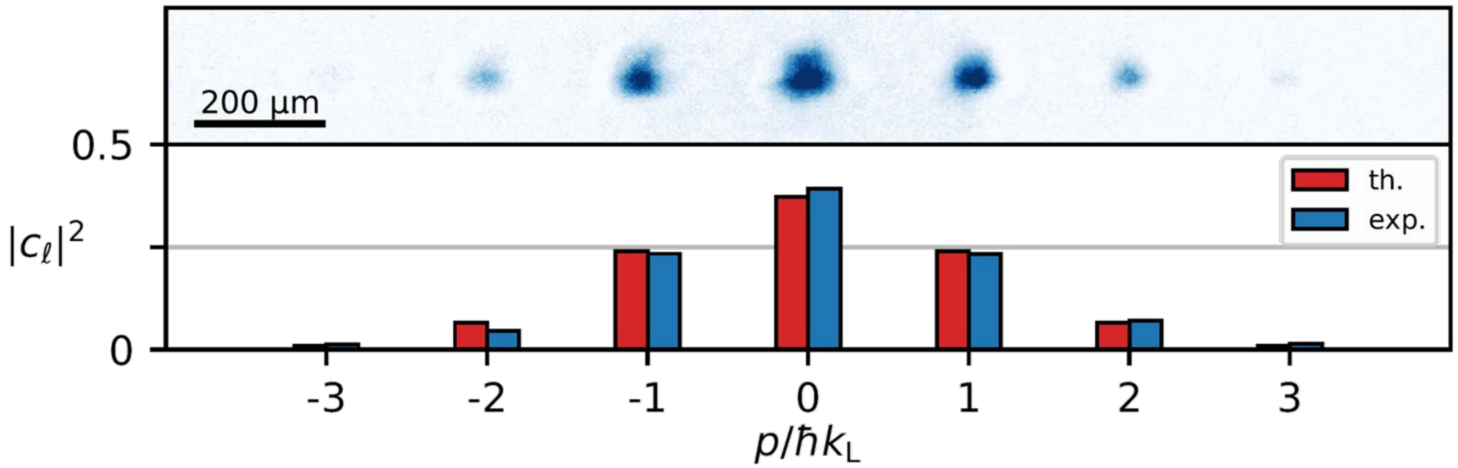
If the initial distribution is small/ the time-of-flight is long

for $t_{TOF} \gg \frac{1}{\omega}$, ω typical trapping frequency,

$$n(\boldsymbol{r}, t = t_{TOF}) = \tilde{n}(\boldsymbol{p} = \frac{m\boldsymbol{r}}{t_{TOF}}, t = 0)$$

We measure the speed (or momentum) distribution!

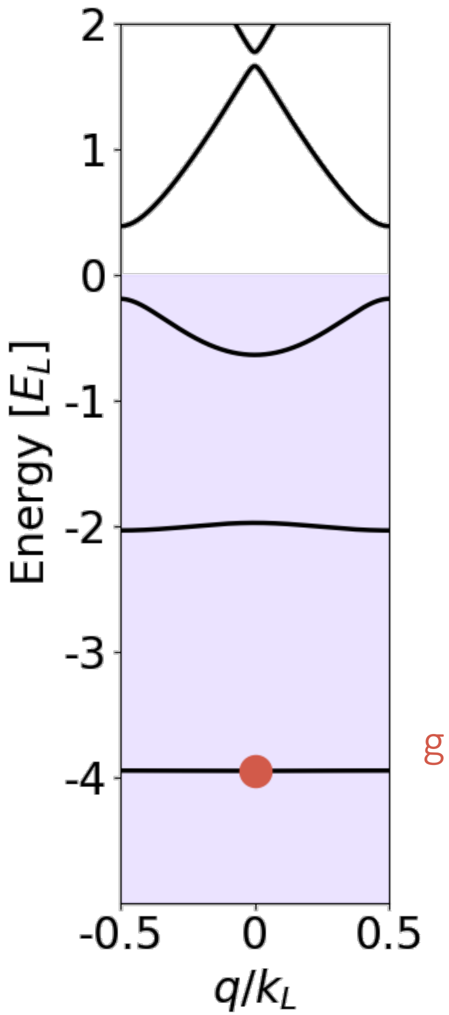
What can we measure?



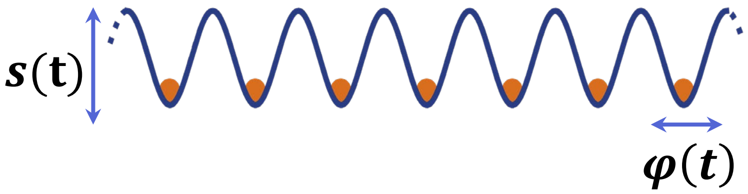
- The momentum distribution gives us exactly the $|c_\ell|^2$ (probabilities)

- Periodic wavefunction in the lattice $\Leftrightarrow$ discrete momentum distribution

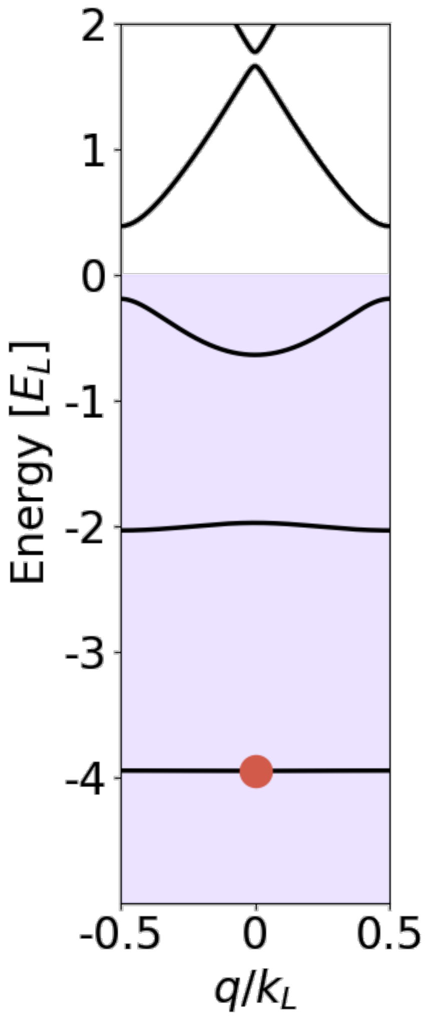
(above: ground state for $s = 20$)



Dynamics



$$V(x,t) = \frac{s(t)E_L}{2}[1 - \cos(k_L x + \varphi(t))]$$



If we start in the ground state, $q = 0$.

The quasi-momentum q is preserved in the dynamics.

→ In the $q = 0$ subspace :
wavefunction expanded on plane waves: $|\psi\rangle = \sum_{\ell\in\mathbb{Z}} c_\ell |\chi_\ell\rangle$

Time-dependent Schrödinger equation?

With $t \to E_\mathrm{L} t/\hbar$

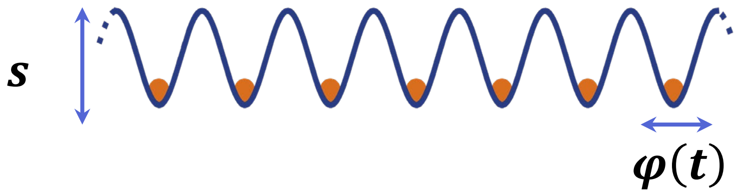
$$i\dot{c}_\ell = \ell^2 c_\ell - \frac{s(t)}{4}\left(e^{i\varphi(t)}c_{\ell-1} + e^{-i\varphi(t)}c_{\ell+1}\right)$$

Question : Can we engineer an arbitrary state,
by tailoring the c_ℓ?

Outline

- BEC in a sine potential
- Optimal control in a sine potential
- State reconstruction
- Control with the non linearity

Optimal control



$$V(x,t) = \frac{s(t)E_L}{2}[1-\cos(k_L x + \varphi(t))]$$

Let's consider control with a phase **(lattice shaking)**:

$$i\dot{c}_\ell = \ell^2 c_\ell - \frac{s}{4}\left(e^{i\varphi(t)}c_{\ell-1} + e^{-i\varphi(t)}c_{\ell+1}\right)$$

$$\Leftrightarrow \; i\dot{C} = \mathcal{M}\left(\varphi(t)\right) \times C$$

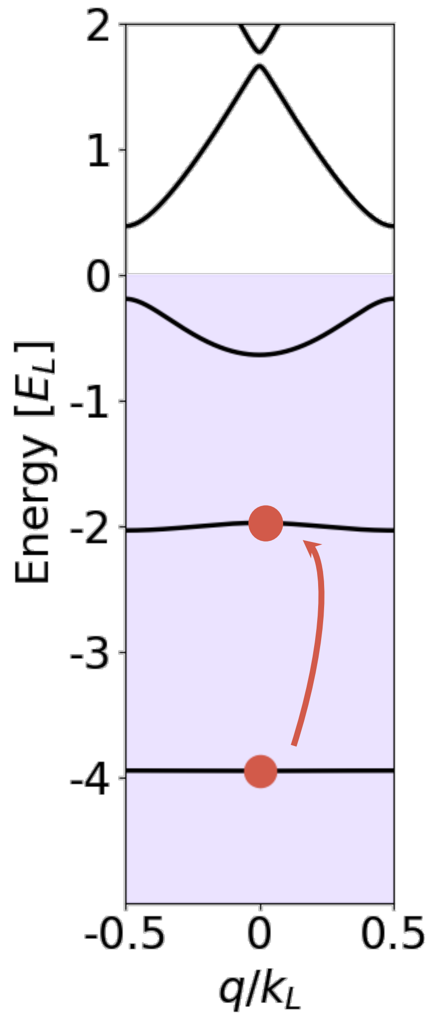
Define a control problem:

→ Control **duration** $t_f \simeq 500\mu s$

→ Control **target** $\boldsymbol{C_T}$ -- we want $C(t_f) \simeq C_T$

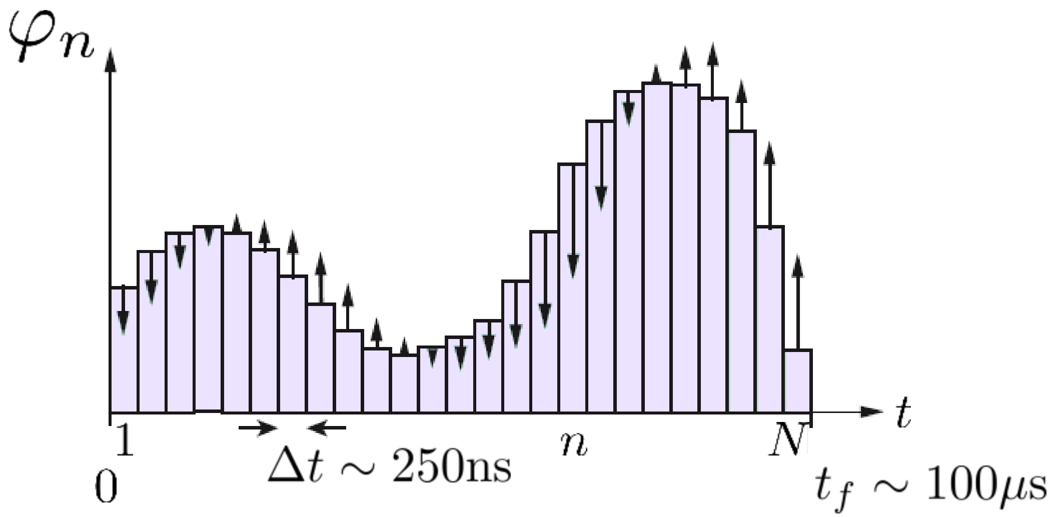
→ **Figure of merit**:

e.g. fidelity $\mathcal{F} = \left|\langle\Psi_T|\Psi(t_f)\rangle\right|^2 = \left|C_T^\dagger C(t_f)\right|^2$



Gradient ascent

- In practice : optimize a discretized phase evolution $\{\varphi_n\}$,



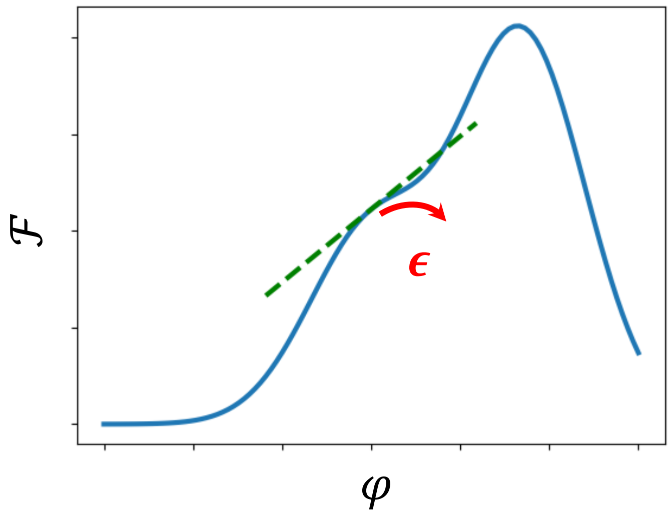
- through **gradient ascent:**

$$\varphi_n^{(k)} \to \varphi_n^{(k+1)} = \varphi_n^{(k)} + \epsilon \frac{\partial \mathcal{F}}{\partial \varphi_n^{(k)}}$$

Iterative process: for small ϵ,

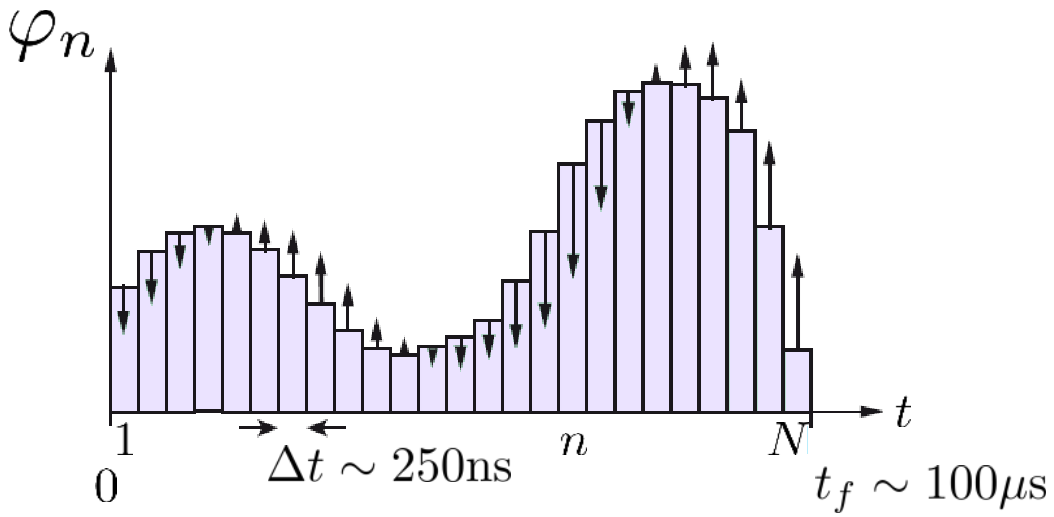
$$\mathcal{F}(\{\varphi_n^{(k+1)}\}) - \mathcal{F}(\{\varphi_n^{(k)}\}) \simeq \epsilon \sum_n \left(\frac{\partial \mathcal{F}}{\partial \varphi_n^{(k)}}\right)^2 > 0$$

→ $\mathcal{F}$ **increases.**



Gradient ascent

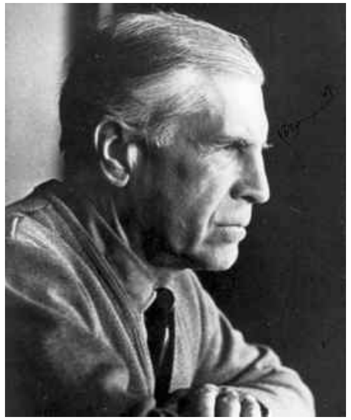
- Can be performed in a concise way using **Pontryiagin's Hamiltonian**



The concept:

In classical dynamics, the **least action principle** gives the equations of motion
→ Hamilton's formulation of mechanics.

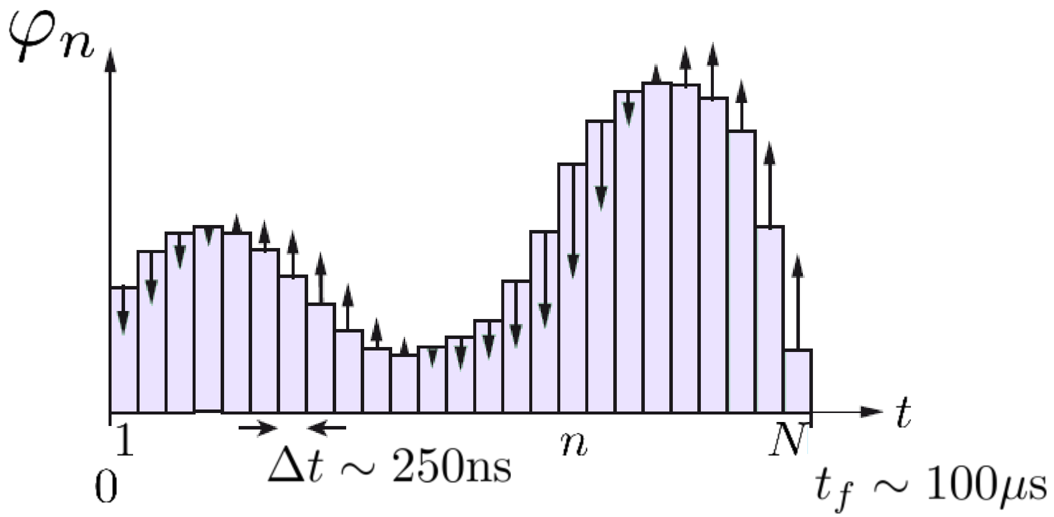
Our extremalization problem can be formulated with an effective **action.** It corresponds to a **Hamiltonian** which must be extremal for the optimal control solution:



$$H_P(C, D, \varphi^*) = \max_{\varphi} H_P(C, D, \varphi^*)$$

Gradient ascent

- Can be performed in a concise way using **Pontryiagin's Hamiltonian**



In practice

For control $\{\varphi_n^{(k)}\}$:

- compute $C(t)$, and the *adjoint* $D(t)$:

$$D(t_f) = \frac{\partial \mathcal{F}}{\partial C^\dagger(t_f)} \quad i\dot{D} = \mathcal{M}\left(\varphi(t)\right) \times D$$

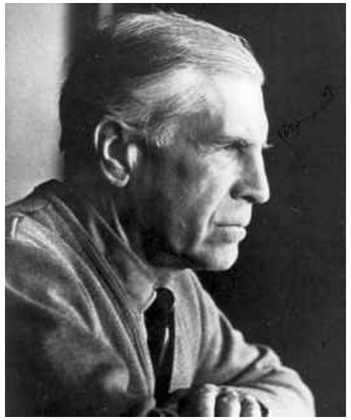
- build *Pontryagin's Hamiltonian*

$$H_{\mathrm{P}} = \mathrm{Re}\left(D^\dagger \dot{C}\right)$$

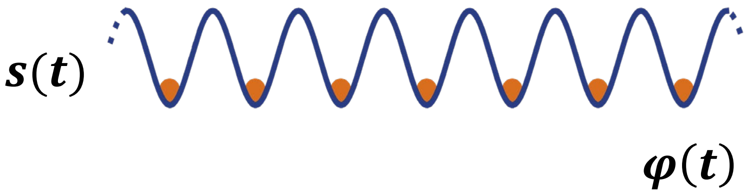
- Apply correction

$$\varphi_n^{(k)} \rightarrow \varphi_n^{(k+1)} = \varphi_n^{(k)} + \epsilon \frac{\partial H_{\mathrm{P}}}{\partial \varphi_n^{(k)}}$$

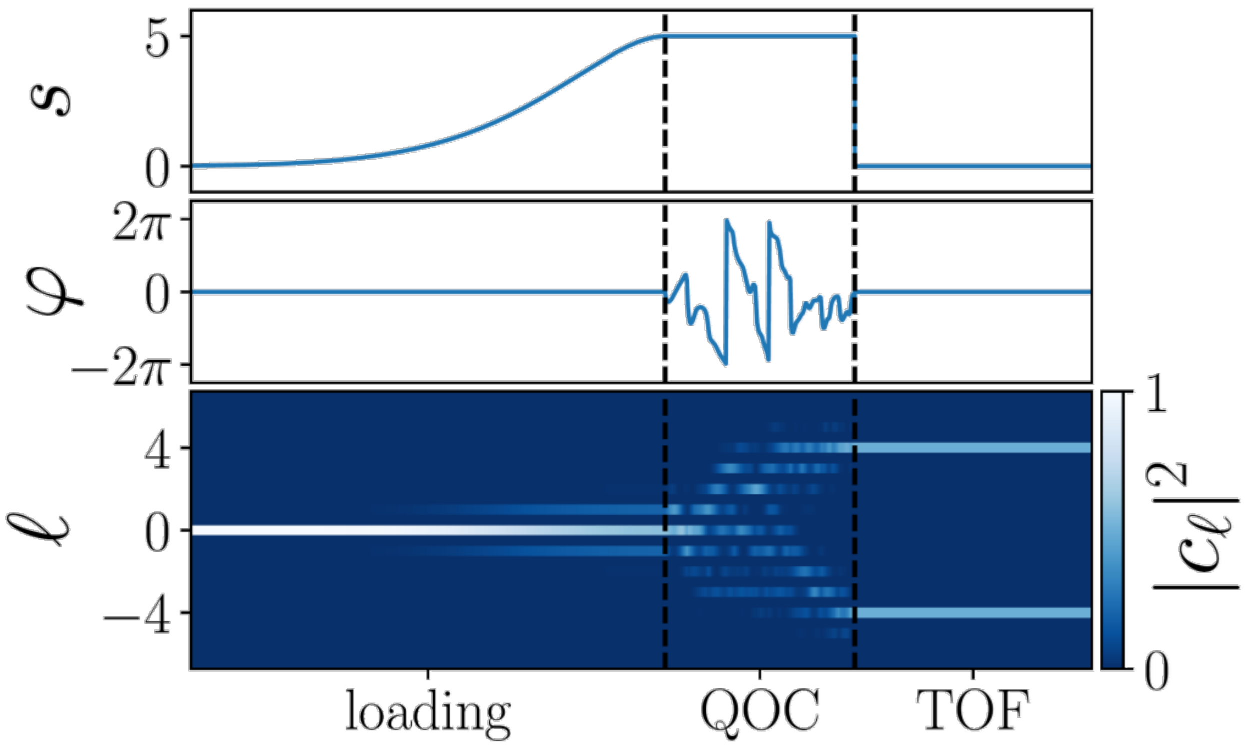
Gradient ascent



Full experimental sequence



- precise lattice depth calibration
- adiabatic lattice loading
- optimal phase control
- time-of-flight, imaging



Simulation of full experimental sequence

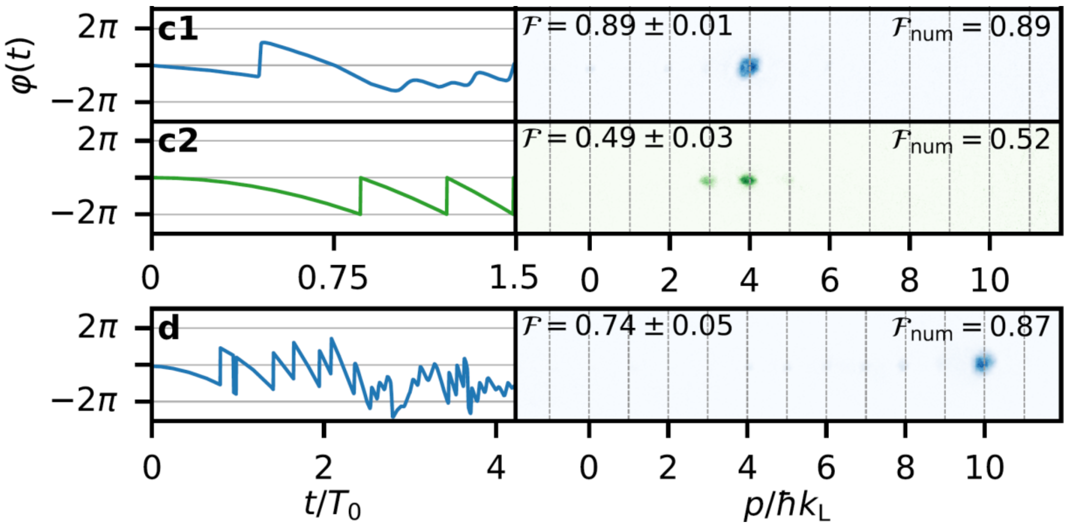
Requires excellent opto-electronic control to implement lattice motion, as well as calibration and stability of s

Control of populations

- Simple figure of merit for probabilities:

$$\mathcal{F} = 1 - \frac{1}{2}\sum_{\ell}(|c_\ell|^2 - p_{t,\ell})^2$$

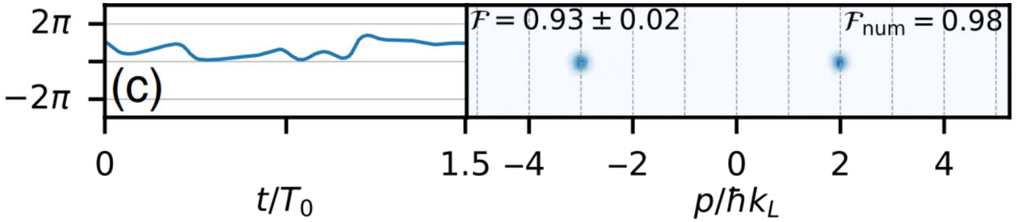
- **Populate** a **specific momentum state**:



Control similar to accelerated lattice – non-adiabatic regime

For **equal performance**, **4-10x faster**

- Population of **multiple components:**



All experiments $s \simeq 5$

$T_0 \simeq 60\mu s$

Lattice trap typical period

N. Dupont *et al*,
PRX *Quantum* **2**, 040303 (2021)

Control of populations

- A robust method, for multiple patterns of populations

All $2^7 = 128$ equal-weights patterns realized!

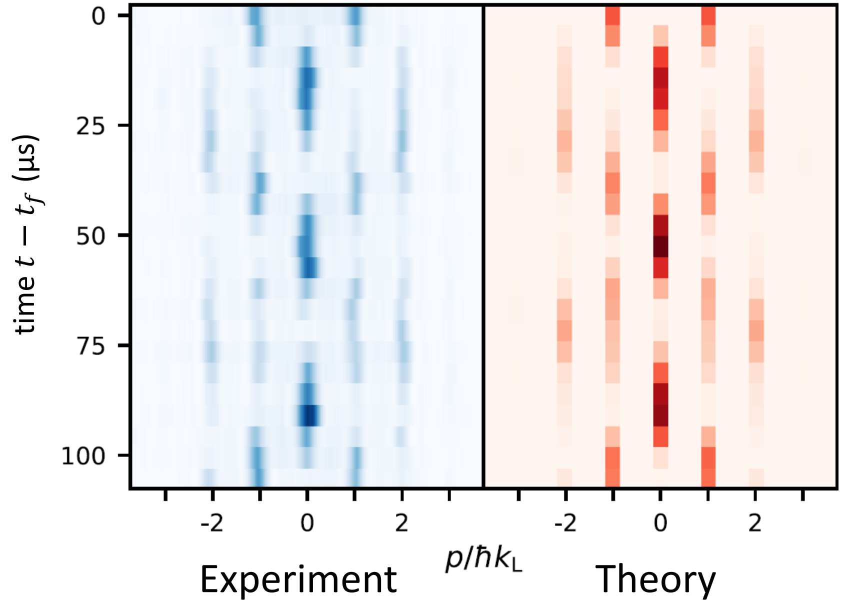
Control of phases

- Figure of merit sensitive to **amplitudes:** $\mathcal{F}_Q = |\langle C_T | C(t_f)\rangle|^2$
- Test on a simple superposition state

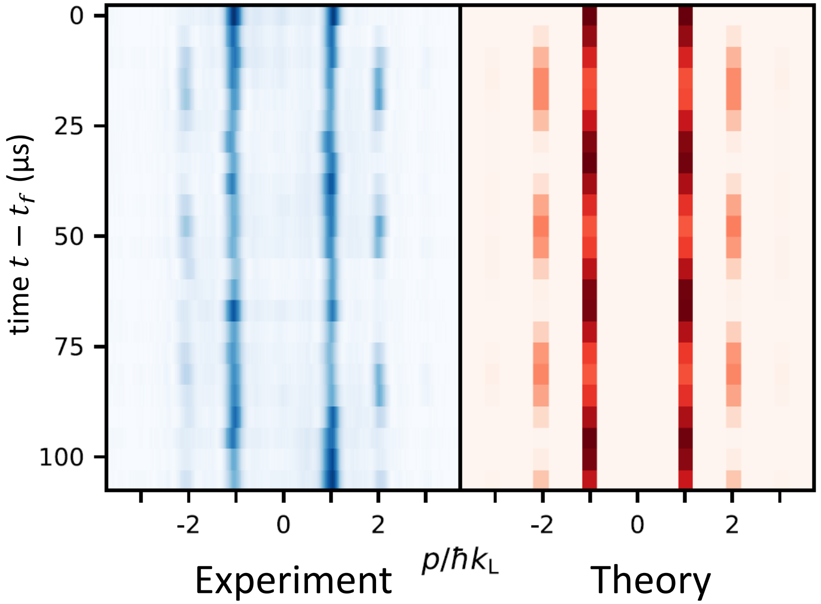
$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\chi_1\rangle + e^{i\Delta\phi}|\chi_{-1}\rangle\right)$$

Identify state from free evolution after preparation:

$$|\psi_T\rangle = \frac{1}{\sqrt{2}}\left(|\chi_1\rangle + |\chi_{-1}\rangle\right)$$



$$|\psi_T\rangle = \frac{1}{\sqrt{2}}\left(|\chi_1\rangle - |\chi_{-1}\rangle\right)$$



PRX Quantum 2, 040303 (2021)

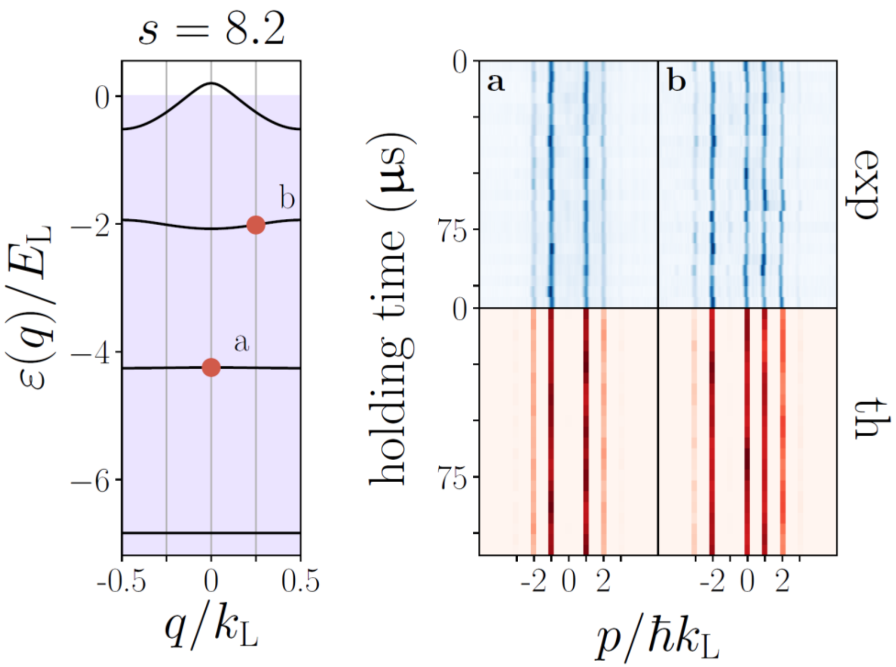
Lattice eigenstates

At quasimomentum q, the n^{th} Bloch function reads

$$|\psi_{n,q}\rangle = \sum_{\ell\in\mathbb{Z}} c_\ell^{(n,q)} |\chi_{\ell+q}\rangle \qquad \chi_\ell(x) \propto e^{i\ell x}$$

With $c_\ell^{(n,q)}$ solutions of the stationary Schrödinger equation

→ We can prepare eigenstates of the lattice potential



The prepared state is stationary for a lattice moving at $v = -\hbar q/m$

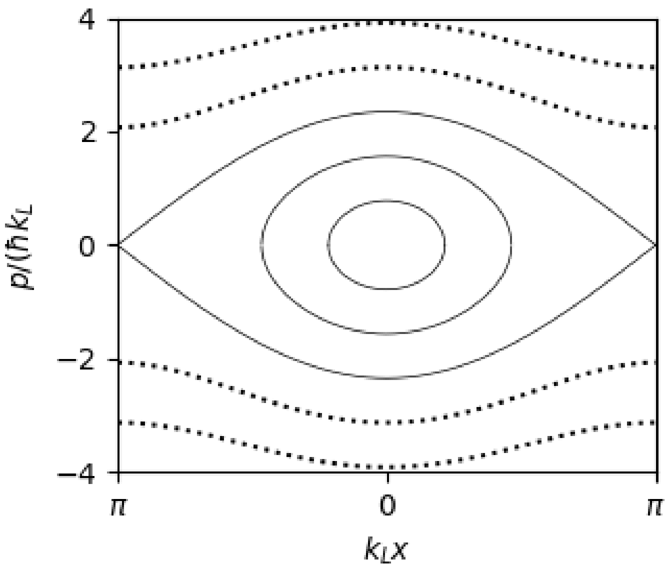
PRX Quantum 2, 040303 (2021)

Outline

- BEC in a sine potential
- Optimal control in a sine potential
- State reconstruction
- Control with the non linearity

States in phase space

- ## Phase space of the lattice potential



The wavefunction is periodic :

identical in each lattice cell from $-\frac{d}{2}$ to $\frac{d}{2}$

→ Phase space with classical trajectories of the pendulum

"Where is the wavefunction?"

Ideally we would like to plot a **probability distribution** over the phase space:

Heisenberg uncertainty principle prevents this!

it's impossible to assign a probability to a single point (x, p)

States in phase space

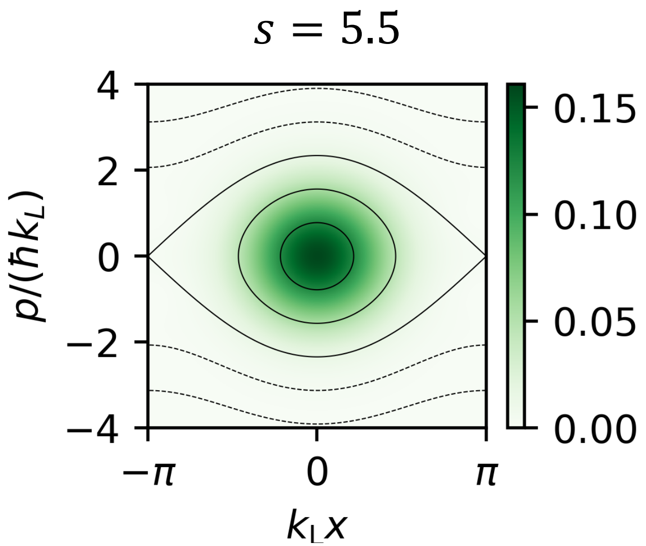
▪ Semiclassical states

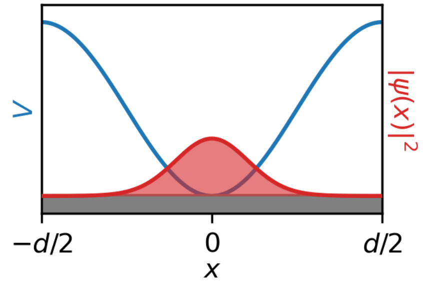
Define a (periodic) **Gaussian state** $|g(u,v)\rangle$ (on plane waves):

$$c_\ell(u,v) \propto e^{-i\ell u} e^{-(\ell-v)^2/\sqrt{s}}$$

- Semiclassical, Heisenberg-limited state centered on $(\langle x\rangle, \langle p\rangle) = (u/k_\mathrm{L}, v \times \hbar k_\mathrm{L})$
- For $s \gg 1$, $|g(0,0)\rangle$ tends to the lattice ground state (gaussian)
- Allows to define a **quasi-distribution (Husimi 1940):**

$$H_{[\rho]}(u,v) \equiv \frac{1}{2\pi}\langle g(u,v)|\rho|g(u,v)\rangle$$

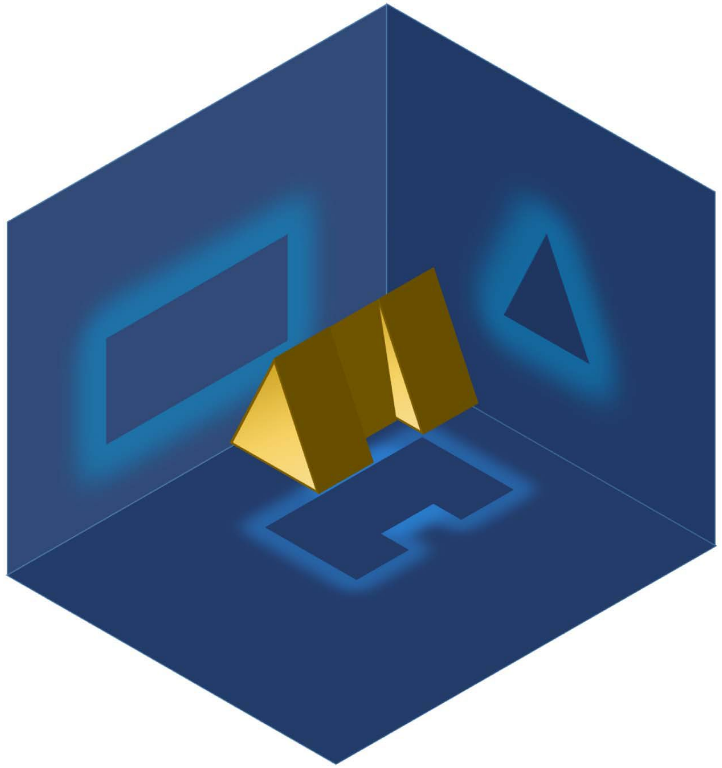


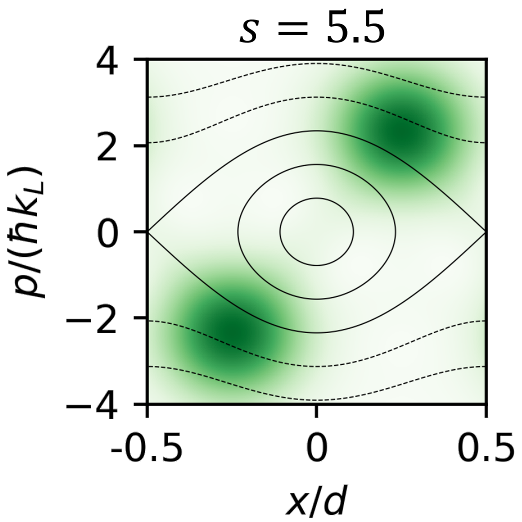


$|g(0,0)\rangle$

States in phase space

- Issue with semiclassical states :
  many momentum components, with many phases
- Requires full **state characterisation/tomography**:

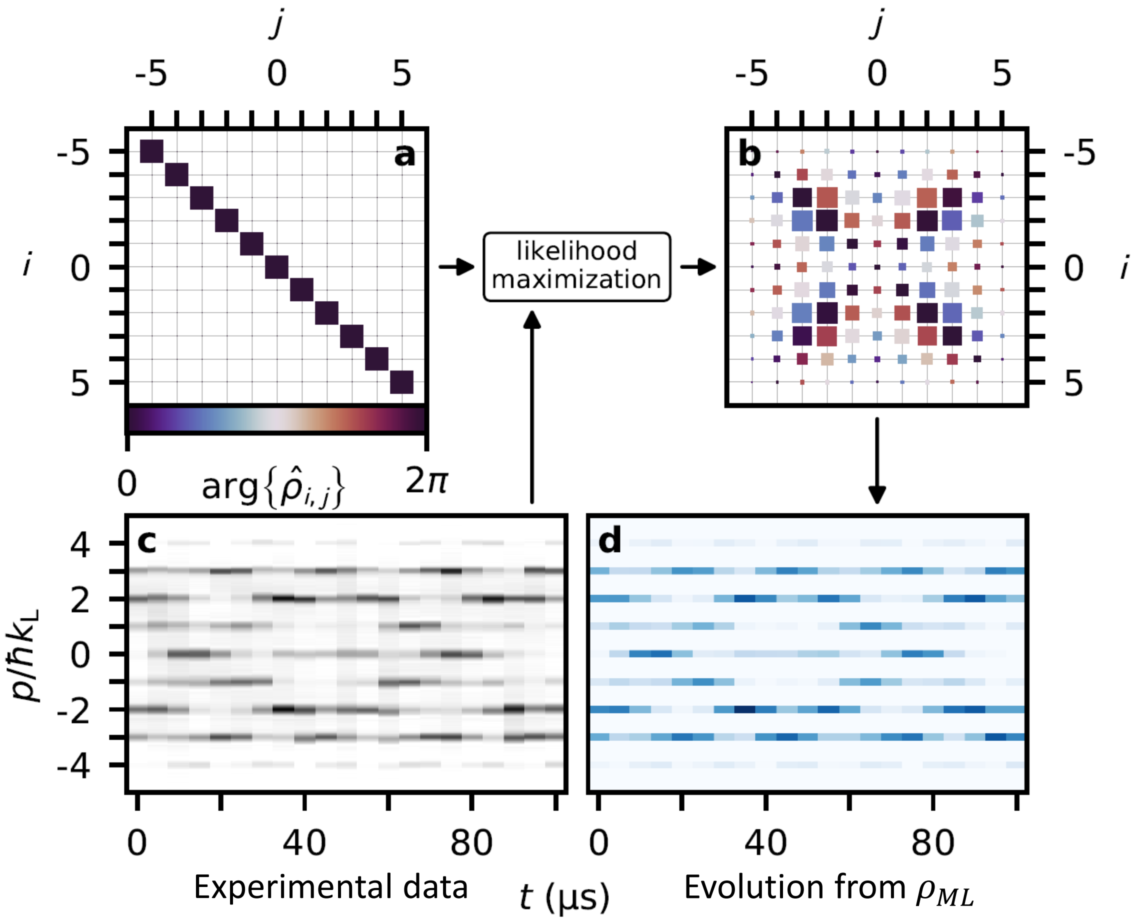


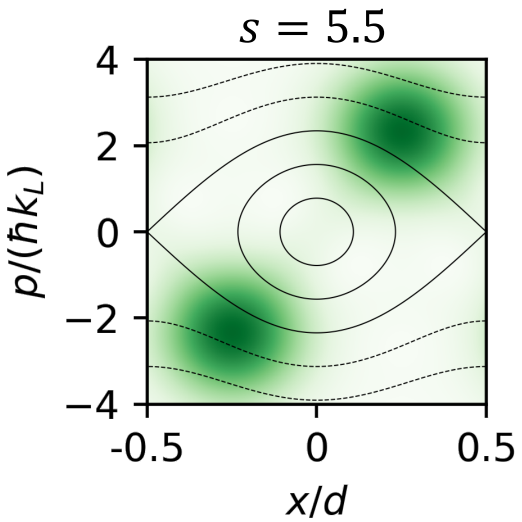


How to reconstruct a complex quantum state from a series of projective measurements?
Exploit "**fingerprint**" from **evolution in static lattice**

States in phase space

- Issue with semiclassical states :
  many momentum components, with many phases
- Requires full **state characterisation/tomography**:





Exploit "**fingerprint**" from **evolution in static lattice** :

Reconstruct a **Maximum Likelihood estimate ρ_{ML}** of the prepared state

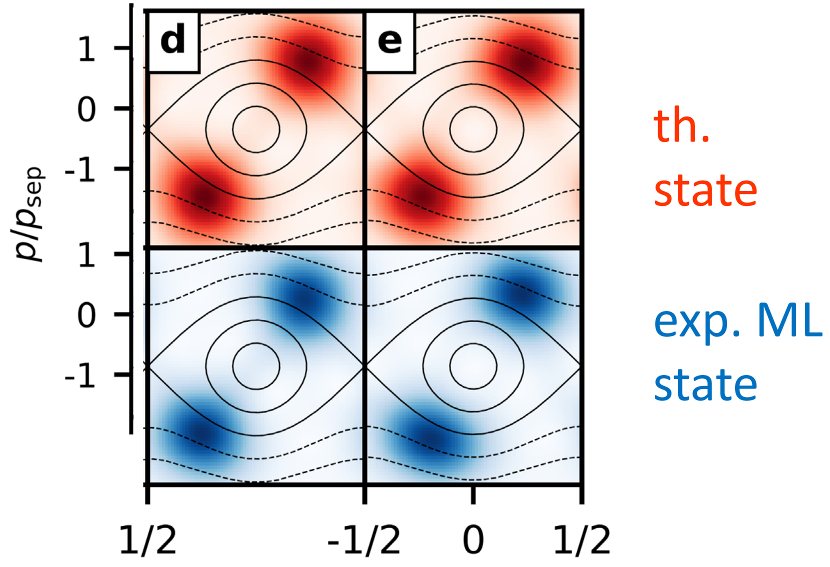
$$\mathcal{L}[\rho] = \prod_i \pi_i^{f_i}$$

Experimental frequency

Theoretical probability

Maximized through an iterative method

Superpositions of Gaussian states



	d	e
u	$\pm\pi/2$	$\pm\pi/2$
v	$\pm\sqrt{s}$	$\pm\sqrt{s}$
$\mathcal{F}_{\rm exp}$	0.89	0.91
γ	0.82	0.91
s	5.5 ± 0.5	5.30 ± 0.25
	even $(+)$	odd $(-)$

- Gaussian state superpositions of opposite parity :

$$|\psi_{\rm even,odd}\rangle \simeq \frac{1}{\sqrt{2}}(|g(u,v)\rangle \pm |g(-u,-v)\rangle),$$

Fidelity between the expected state and the reconstructed ML state:

$$\mathcal{F}_{\rm exp} = \langle\psi_{\rm th}|\rho_{\rm ML}|\psi_{\rm th}\rangle$$

Purity of the reconstructed ML state:

$$\gamma = {\rm Tr}(\rho_{\rm ML}^2)$$

Purity is affected by fluctuations between measurements

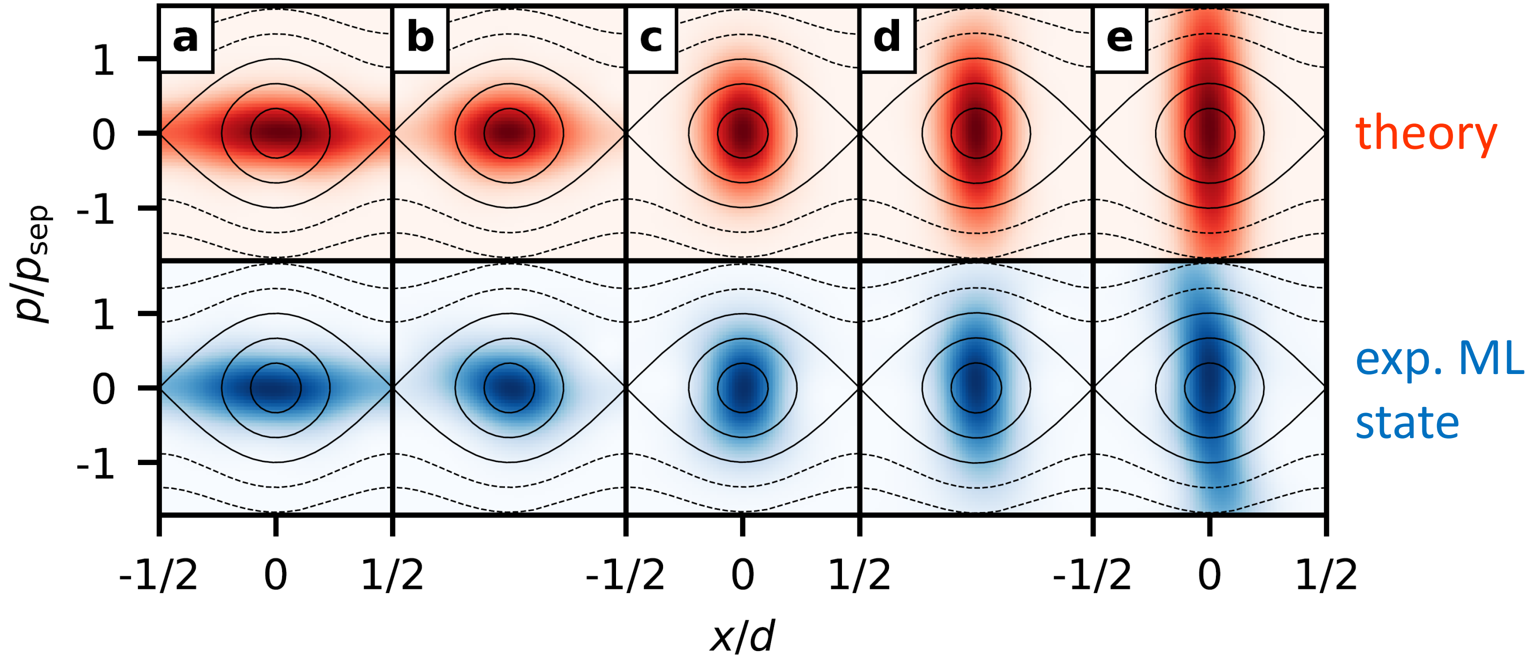
Squeezed states

- **Squeezed state :** modified variances with respect to the ground state

$$\xi = \Delta x(\xi)/\Delta x^{(g)} = \left(\Delta p(\xi)/\Delta p^{(g)}\right)^{-1}$$

$\xi < 1 \rightarrow$ position squeezing

N. Dupont *et al.*, *New J. Phys.* **25**, 013012 (2023)

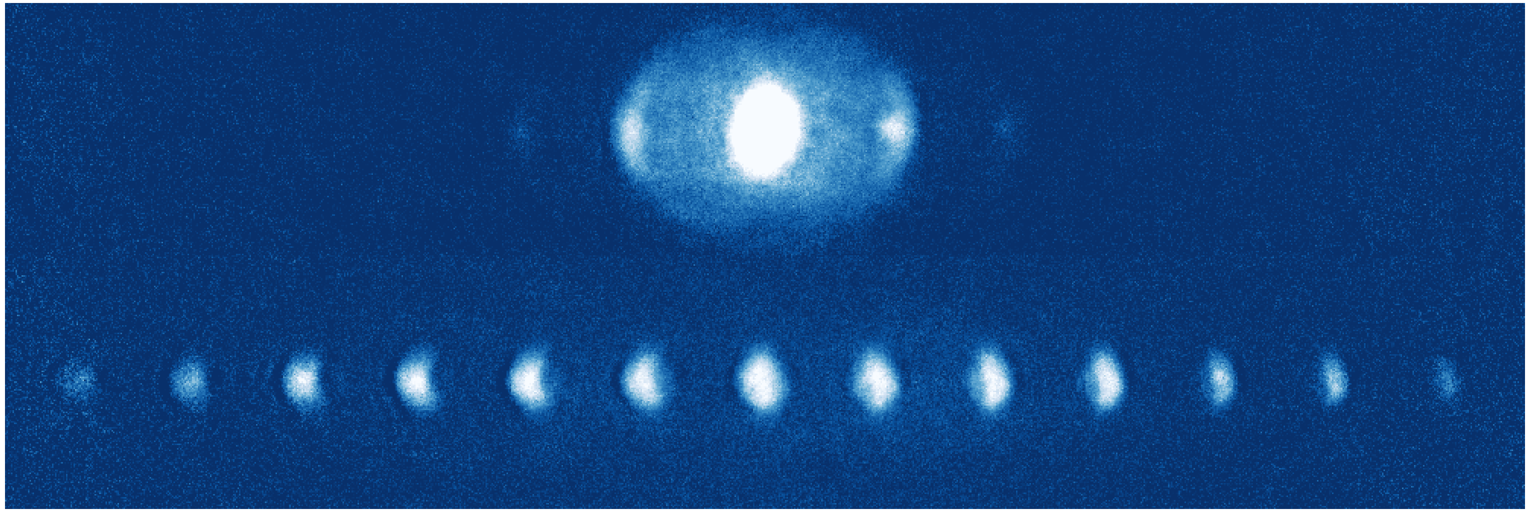


$1/\xi$	0.44	0.62	1.65	2.75	4.34
$\mathcal{F}_{\text{exp}}$	0.99	0.96	0.98	0.93	0.75
γ	1.00	1.00	1.00	0.92	0.72

Extreme squeezing

- A **squeezed state** with parameter ξ is **similar** to the **ground state** of a lattice with **effective depth:**

$$s_{\mathrm{eff}} = s/\xi^4$$

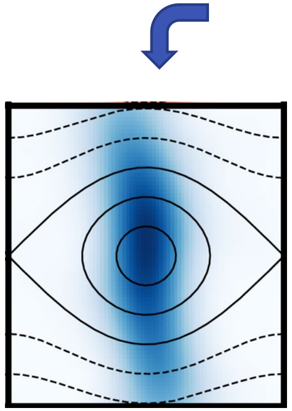


Ground state, $s = 5.6$

Squeezed state

$s = 5.6,\ 1/\xi = 4.34$

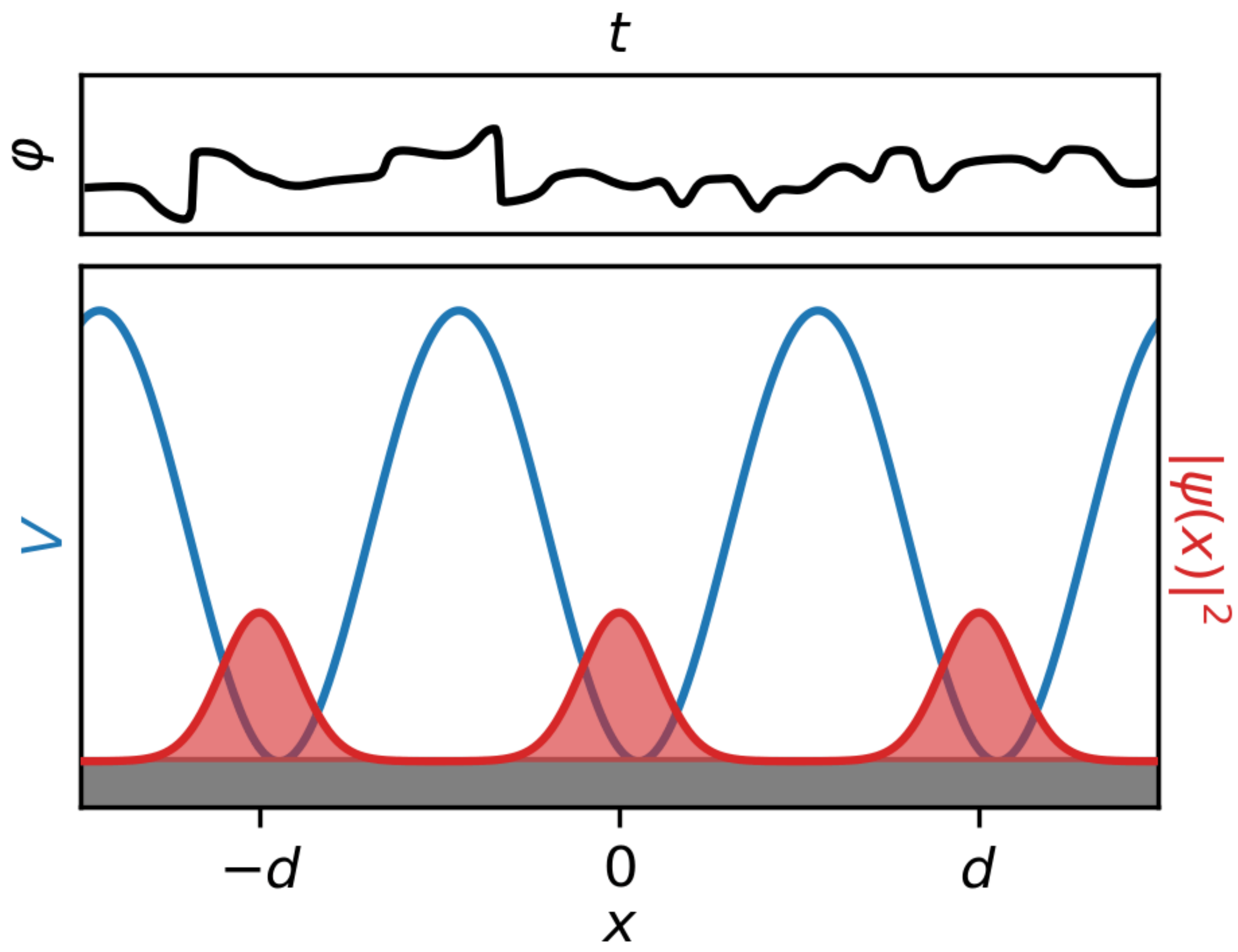
$s_{\mathrm{eff}} \simeq 2000$



Preparation of a **technically inaccessible** state
on **short timescale compared to adiabatic methods**

Extreme squeezing

- **Evolution during preparation (theory)**



N. Dupont *et al.*, *New J. Phys.* **25**, 013012 (2023)

Squeezing in position space equivalent to a x400 laser intensity increase !

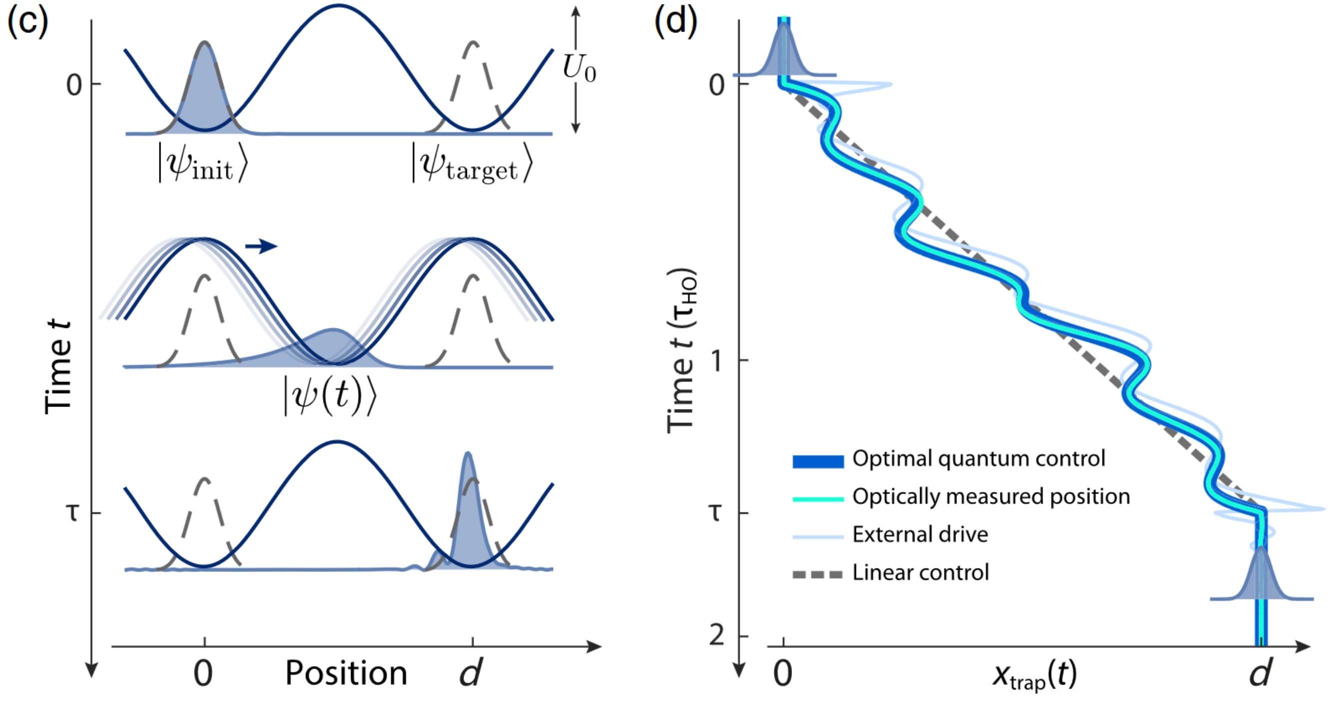
Aside: brachistochrone

PHYSICAL REVIEW X **11**, 011035 (2021)

Featured in Physics

Demonstration of Quantum Brachistochrones between Distant States of an Atom

Manolo R. Lam,[1] Natalie Peter,[1] Thorsten Groh,[1] Wolfgang Alt,[1] Carsten Robens,[1,2] Dieter Meschede,[1] Antonio Negretti,[3] Simone Montangero,[4] Tommaso Calarco,[5] and Andrea Alberti[1,*]



The same control logic can be applied to transport problems!

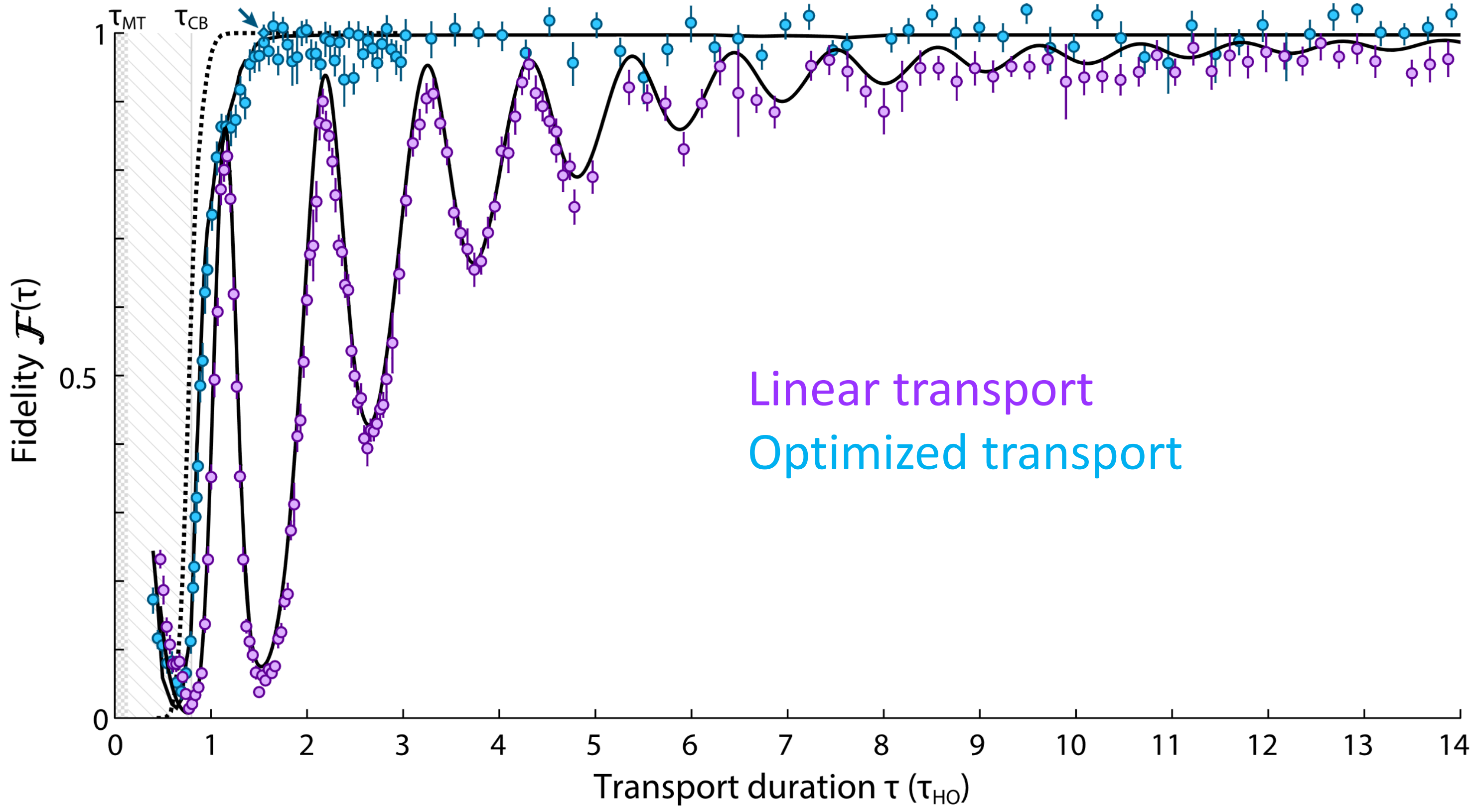
Aside: brachistochrone

PHYSICAL REVIEW X **11,** 011035 (2021)

Featured in Physics

Demonstration of Quantum Brachistochrones between Distant States of an Atom

Manolo R. Lam,[1] Natalie Peter,[1] Thorsten Groh,[1] Wolfgang Alt,[1] Carsten Robens,[1,2] Dieter Meschede,[1] Antonio Negretti,[3] Simone Montangero,[4] Tommaso Calarco,[5] and Andrea Alberti[1,*]



The same control logic can be applied to transport problems!

Outline

- BEC in a sine potential
- Optimal control in a sine potential
- State reconstruction
- Control with the non linearity

Non-linearity

Recall that the atoms are **interacting:**

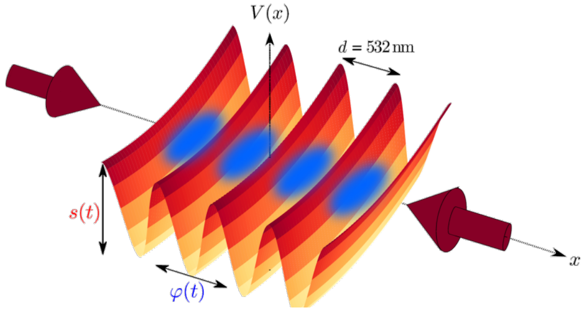
we can account for interactions at the mean-field level with the **Gross-Pitaevskii equation**

$$i\hbar \frac{d}{dt}\psi(r,t) = \left[-\frac{\hbar^2}{2m}\Delta + V(r) + Ng|\psi(r)|^2\right]\psi(r,t)$$

Non-linearity

Recall that the atoms are **interacting:**
we can account for interactions at the mean-field level with the **Gross-Pitaevskii equation**

$$i\hbar \frac{d}{dt}\psi(r,t) = \left[-\frac{\hbar^2}{2m}\Delta + V(r) + Ng|\psi(r)|^2\right]\psi(r,t)$$



In our 1D potential, **approximation**:
integrate out slow transverse dynamics

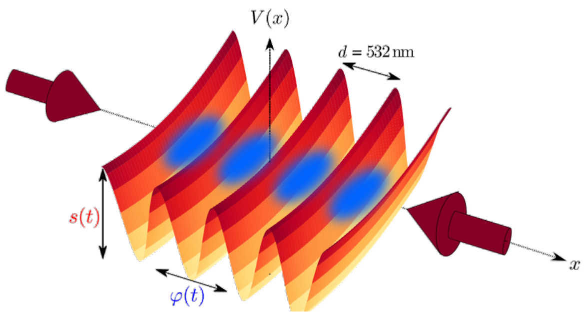
$$i\frac{d}{dt}\psi(x,t) = \left[-\frac{d^2\psi(x,t)}{dx^2} - \frac{s(t)}{2}\cos(x+\varphi) + \beta|\psi(r)|^2\right]\psi(r,t)$$

β: effective 1D interaction strength ($\beta \sim 1$)

Non-linearity

Recall that the atoms are **interacting:**
we can account for interactions at the mean-field level with the **Gross-Pitaevskii equation**

$$i\hbar \frac{d}{dt}\psi(r,t) = \left[-\frac{\hbar^2}{2m}\Delta + V(r) + Ng|\psi(r)|^2\right]\psi(r,t)$$



In our 1D potential, **approximation**:
integrate out slow transverse dynamics

$$i\frac{d}{dt}\psi(x,t) = \left[-\frac{d^2}{dx^2} - \frac{s(t)}{2}\cos(x+\varphi) + \beta|\psi(r)|^2\right]\psi(r,t)$$

β: effective 1D interaction strength ($\beta \sim 1$)

Dynamics in the zero quasi-momentum subspace:

$$i\dot{c}_\ell = \ell^2 c_\ell - \frac{s}{4}\left(e^{i\varphi(t)}c_{\ell-1} + e^{-i\varphi(t)}c_{\ell+1}\right) + \frac{\beta}{2\pi}\sum_{m,n\in\mathbb{Z}} c_m^* c_n c_{n+m-\ell}$$

non-linear term:
many contributions

Non-linearity

$$i\dot{c}_\ell = \ell^2 c_\ell - \frac{s}{4}\left(e^{i\varphi(t)}c_{\ell-1} + e^{-i\varphi(t)}c_{\ell+1}\right) + \frac{\beta}{2\pi}\sum_{m,n\in\mathbb{Z}} c_m^* c_n c_{\ell-n+m}$$

Non-linearity → numerical step-wise integration:
$U(T) = U((M-1)\delta t \to M\delta t) \times \cdots U(\delta t \to 2\delta t) \times U(0 \to \delta t)$ with M large.
Options :
1) brute force approach (e.g Runge-Kutta),
2) take advantage of the structure:

$$H(t) = H_0 + H_{int}$$

"simple" in momentum space (c_ℓ) ↗ H_0

"simple" in position space $\psi(x) = \sum_\ell \frac{c_\ell}{\sqrt{2\pi}} e^{i\ell x}$ ↖ H_{int}

Trotter approximation:

$$U(\delta t) \simeq \exp(-iH_{int}(t)\delta t)\exp(-iH_0(\varphi)\delta t)$$

Non-linearity

$H_{int} = \beta \int dx \, |\psi(x,t)|^2 |x\rangle\langle x|$ is diagonal in position.

→ Change basis from momentum basis $|\ell\rangle$ (N plane waves) to an approximate, **discrete position basis** of the lattice cell $x \in [0,2\pi]$:

$$\left| u_j \left(x_j = \frac{2\pi}{N} j \right) \right\rangle = \sum_\ell \frac{e^{-i\ell\frac{2\pi}{N}j}}{\sqrt{N}} |\ell\rangle$$

- $\langle u_j|\psi\rangle \simeq \sqrt{\frac{2\pi}{N}} \psi(x_j) = \sum_j \frac{c_\ell e^{i\ell\frac{2\pi}{N}j}}{\sqrt{N}}$ is the **discrete Fourier transform** of the c_ℓ
- We can represent an operator $W(\hat{x})$ as $W(\hat{x}) \simeq \sum_j W(x_j) \, |u_j\rangle\langle u_j|$

$$C = \begin{pmatrix} \cdots \\ c_{\ell-1} \\ c_\ell \\ c_{\ell+1} \\ \cdots \end{pmatrix} \xrightarrow[\text{basis change}]{\hat{R}} \psi_u = \begin{pmatrix} \cdots \\ \langle u_{j-1}|\psi\rangle \\ \langle u_j|\psi\rangle \\ \langle u_{j+1}|\psi\rangle \\ \cdots \end{pmatrix} \longrightarrow \psi'_u \xrightarrow[\text{basis change}]{\hat{R}^\dagger} C' = e^{-iH_{int}(t)\delta t} C$$

$$R_{j,\ell} = \frac{e^{i\ell\frac{2\pi}{N}j}}{\sqrt{N}}$$

$$\exp(-iH_{int}(t)\delta t) \simeq \sum_j \exp\left(-i\beta \left|\psi(x_j)\right|^2 \delta t\right) |u_j\rangle\langle u_j|$$

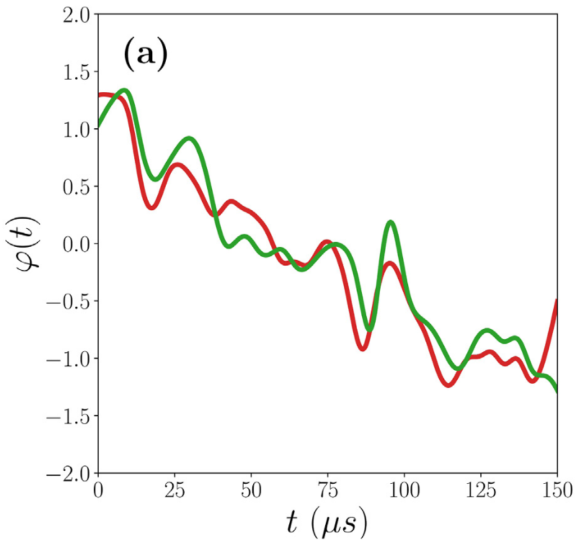
Non-linearity

$$U(\delta t) \simeq \exp(-iH_{int}(t)\delta t)\exp(-iH_0(\varphi)\delta t)$$

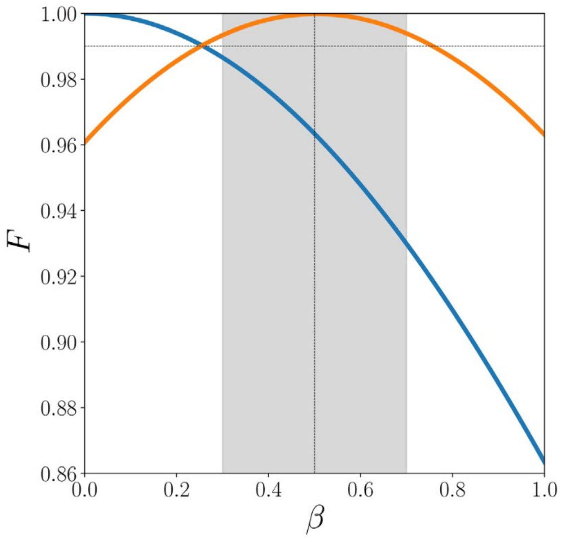
Compute in $|u_j\rangle$ basis — Compute in $|\ell\rangle$ basis

→ Simplifies the matrix exponential for the interactions
→ Gradient ascent can be adapted for optimal control with interactions

Example for moderate interactions $\beta = 0.5$, preparation of a squeezed state:



ramp optimized without interactions
ramp optimized with interactions

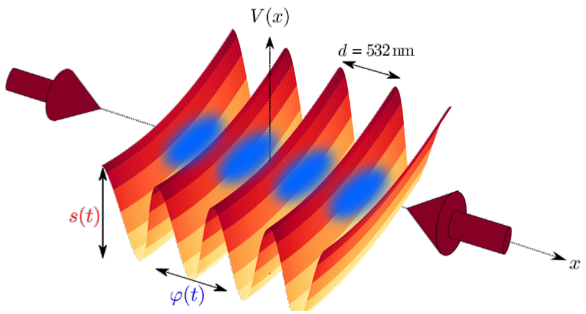


ramp optimized without interactions
ramp optimized with interactions

Limited impact on experimental data for realistic values of β

E. Dionis *et al.*,
Front. Quantum Sci. Technol. **4**: 1540695 (2025)

Conclusion



- Optimal control can be applied to a BEC in a lattice for efficient state-to-state transfer
- We can assess the quality of the result by state reconstruction
- Interactions can be taken into account

- Multiple uses:

Quantum Optimal Control

Quantum Simulation

Lecture III

Optimal sensing

N. Ombredane

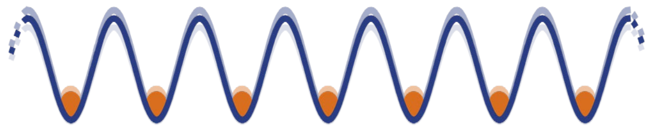
Quantum gates

$$\begin{cases} |\psi_1\rangle \\ |\psi_2\rangle \\ |\psi_3\rangle \\ \dots \end{cases} \xrightarrow{\hat{U}(\varphi(t))} \begin{cases} |\psi'_1\rangle \\ |\psi'_2\rangle \\ |\psi'_3\rangle \\ \dots \end{cases}$$

E. Flament

Extensions

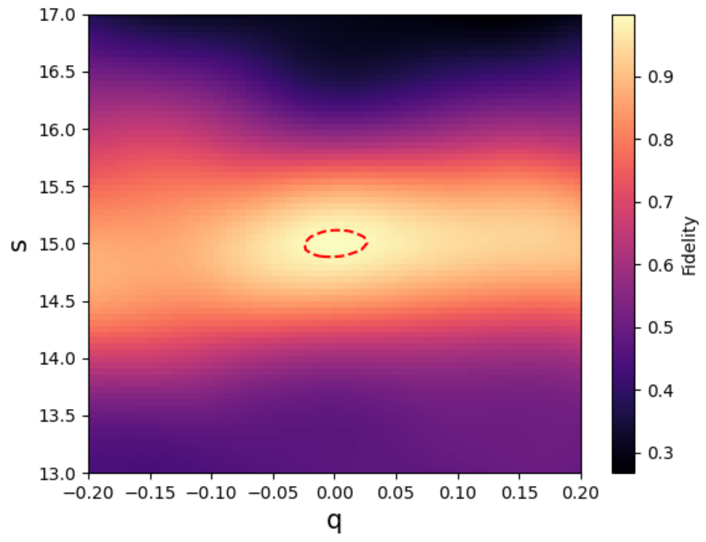
Instead of enhanced *sensitivity* (for parameter estimation)
we may want to be *robust* against the variation of a parameter



If s fluctuates from experiment to experiment,
or to account for the finite size → width Δq

- Select ensemble of discrete values $\{s_i\}, \{q_j\}$
- Modify the Gradient Ascent:
  at each iteration k, calculate all the corrections
  $$\delta\varphi_{n,i,j}^{(k)} = \frac{\partial \mathcal{F}_{i,j}}{\partial \varphi_n^{(k)}}$$
  where $\mathcal{F}_{i,j}$ is computed from the evolution with fixed (s_i, q_j)
- Modify φ for the next iteration with the **average correction:**

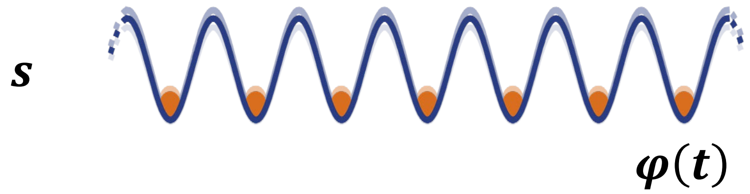
$$\varphi_n^{(k+1)} = \varphi_n^{(k)} + \epsilon \left\langle \delta\varphi_{n,i,j}^{(k)} \right\rangle_{i,j}$$



Fidelity map in {s,q} for non-robust preparation of a squeezed state

Extensions

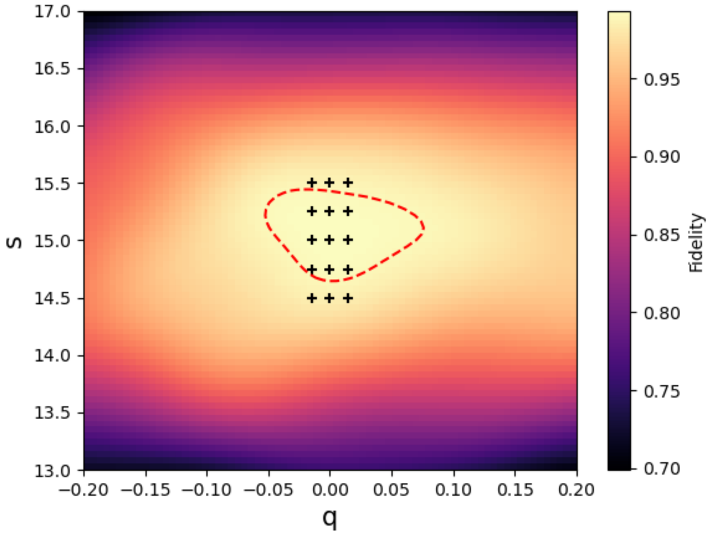
Instead of enhanced *sensitivity* (for parameter estimation)
we may want to be *robust* against the variation of a parameter



If s fluctuates from experiment to experiment,
or to account for the finite size → width Δq

- Select ensemble of discrete values $\{s_i\}, \{q_j\}$
- Modify the Gradient Ascent:
  at each iteration k, calculate all the corrections
  $$\delta\varphi_{n,i,j}^{(k)} = \frac{\partial \mathcal{F}_{i,j}}{\partial \varphi_n^{(k)}}$$
  where $\mathcal{F}_{i,j}$ is computed from the evolution with fixed (s_i, q_j)
- Modify φ for the next iteration with the **average correction:**

$$\varphi_n^{(k+1)} = \varphi_n^{(k)} + \epsilon \left\langle \delta\varphi_{n,i,j}^{(k)} \right\rangle_{i,j}$$



Fidelity map in {s,q} for **robust** preparation of a squeezed state

The end