QUANTUM
ENGINEERING
TOULOUSE

in
collaboration:

Experiments with Bose-Einstein condensates

From atom cooling to quantum control and simulation

QuanTEEM Winter School
Dijon, 24/02/2025

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Lecture I

Atom cooling

Introduction

• A long road to Bose-Einstein condensation

Satyendranath Bose
1894-1974

Albert Einstein
1879-1955

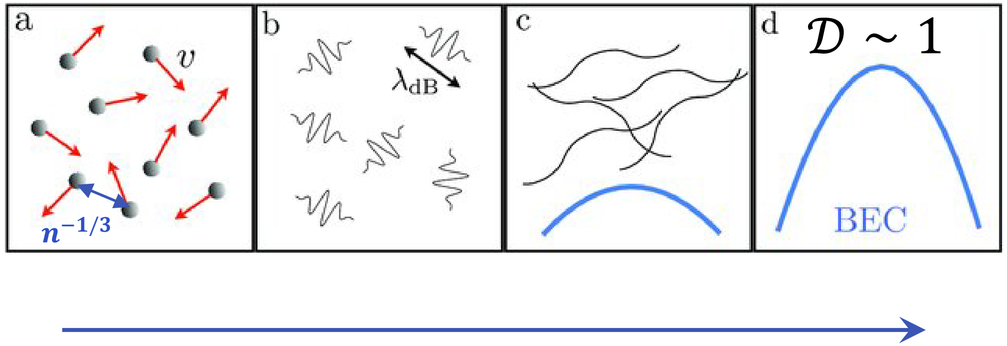
In 1925, Einstein extends the derivation of quantum statistics used by Bose (1924) to obtain Planck's law of blackbody radiation, and applies it to the ideal gas.
He notes that, at fixed T, a density increase will lead to a *macroscopic population of the ground state.*

This phase transition is called **Bose-Einstein condensation**.

In practical terms, what matters is the inter-particle distance vs de Broglie wavelength

Phase space density: $\mathcal{D} = n\lambda_T^3$

$$\lambda_T = \hbar\sqrt{\frac{2\pi}{mk_BT}}$$



decrease temperature

Introduction

- **A technical challenge**

BEC is a property of the *ideal* gas (non-interacting):
In most systems, interactions are non-negligible as density increases, and other phase transition (liquefaction, deposition) may occur.

→ Requires working with dilute gases !
First successful attempts on alkali atom gases

$$\mathcal{D} = n\lambda_T^3 \propto nT^{-3/2}$$

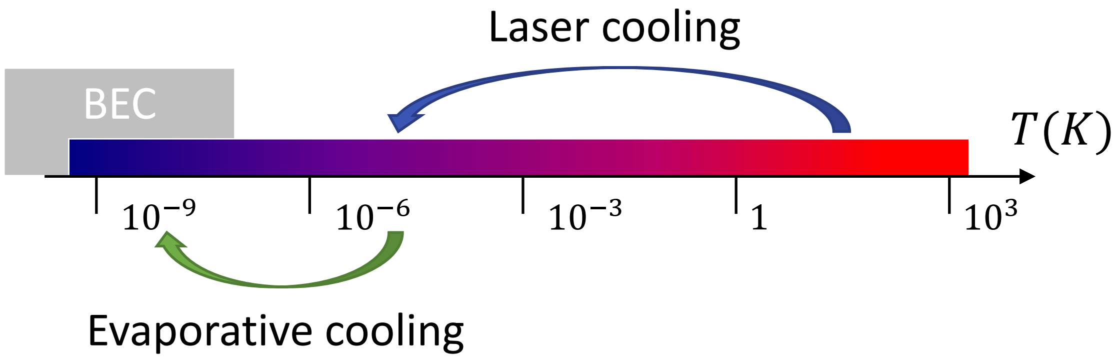
Typical densities
$n \simeq 10^{12} - 10^{13} cm^{-3}$

For ^{87}Rb
$\mathbf{T \simeq 100}\boldsymbol{nK}$

- How do we confine atoms?
- How do we cool them?
- Without direct contact with the room temperature?

Introduction

- **A technical challenge**


Laser cooling

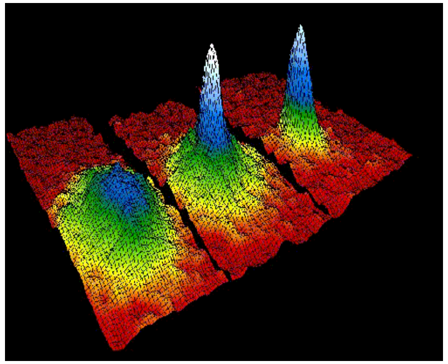
BEC

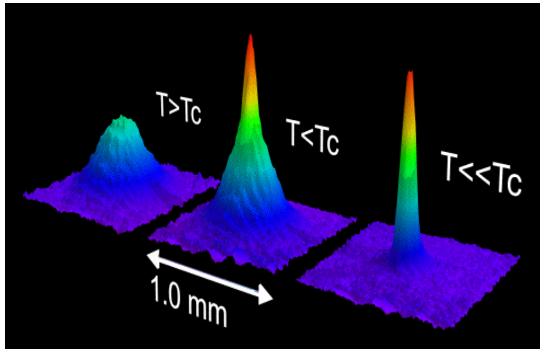
$T(K)$

Evaporative cooling

Progress in laser technology finally lead to the first observation of BEC in 1995:

JILA

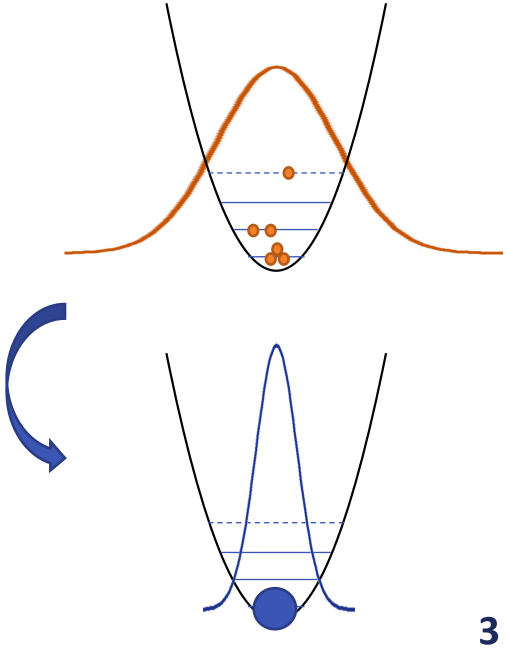

Science, 269, 198 (1995)



Phys. Rev. Lett. 75, 3969 (1995)

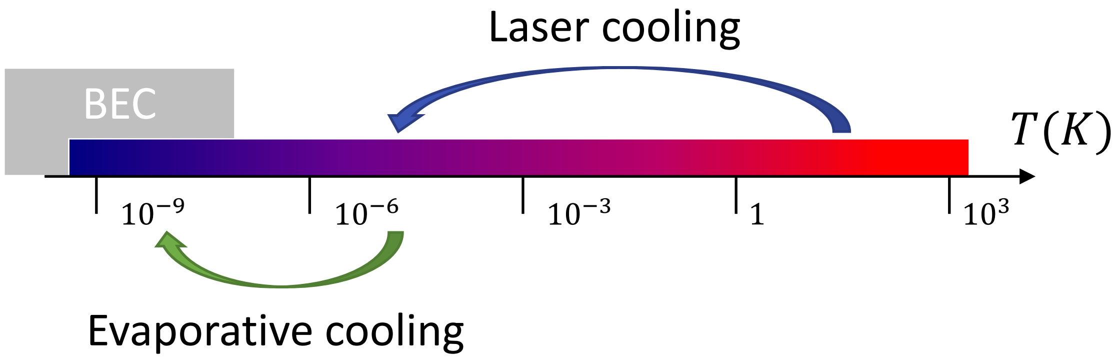
MIT

From a thermal distribution on levels


To most atoms in the ground state

Introduction

- **A technical challenge**



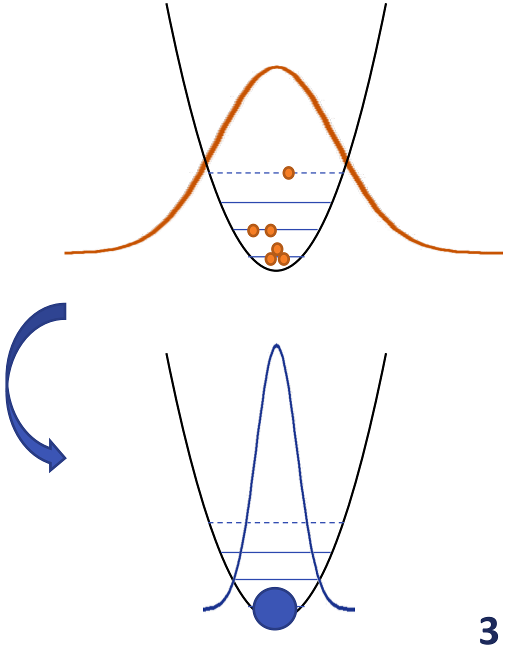
Progress in laser technology finally lead to the first observation of BEC in 1995:

E. Cornell
JILA

W. Ketterle
MIT

C. Wieman
JILA

Nobel Prize 2001



Outline

- Basics of laser (Doppler) cooling
- Evaporative cooling – towards condensation
- The condensation transition
- Interacting condensates – the Gross-Pitaevskii equation

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Laser cooling

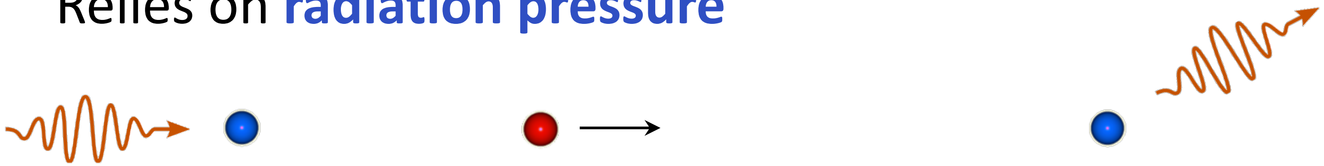
- **Reduce the temperature** of an atomic gas
  → reduce the velocity dispersion

$$\boxed{\frac{1}{2}m(\Delta v)^2 = \frac{3}{2}k_B T}$$

Energy equipartition

Typical value: T=300K, m_{Rb} $\Delta v \sim 300\mathrm{m/s}$

- Cooling = **reduce the speed** of the atoms
  Relies on **radiation pressure**



$\hbar\boldsymbol{k}$

$\Delta\boldsymbol{p} = \hbar\boldsymbol{k}$

$\langle\Delta\boldsymbol{p}\rangle = 0$

Characteristic change in speed

$$v_{rec} = \frac{\hbar k}{m}$$

recoil velocity ($6\ \mathrm{mm/s}$ for Rb)

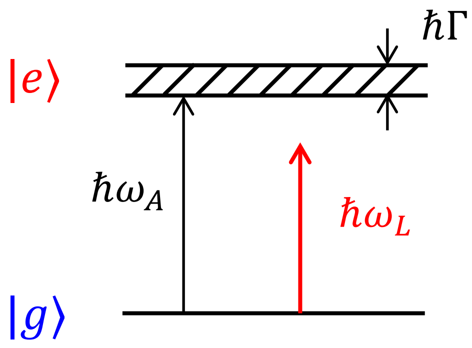
An atom absorbing a photon
gets a change in momentum

De-excitation of the atom:
photon emitted in a random direction

On average, absorption-emission with a beam of light
modifies the speed of the atom

Two-level model

- **Two-level model:** (simplified model of atomic transition)



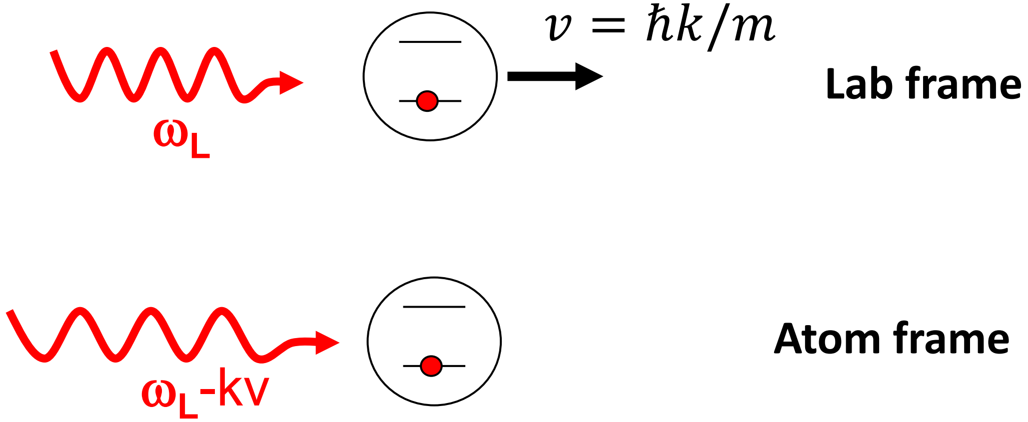
Γ = natural linewidth = $1/\tau_R$

τ_R = radiative lifetime of the excited state (~ 10^{-8} s)

ω_L = laser angular frequency

An atom in the excited state decays with rate $\Gamma = 1/\tau_R$

Two timescales: τ_R, the radiative lifetime
τ_{ext}, timescale of change for external variables



The atom with recoil is ***detuned*** by

$$\delta\omega \simeq kv = \hbar k^2/m$$

The detuning condition is changed for

$$\Gamma\tau_{ext}\delta\omega \sim \Gamma$$

$$\boldsymbol{\tau_{ext} \sim \frac{1}{\delta\omega} \sim \hbar/E_{rec}}$$

$E_{rec} = \frac{\hbar^2 k^2}{2m}$: recoil energy

Two-level model

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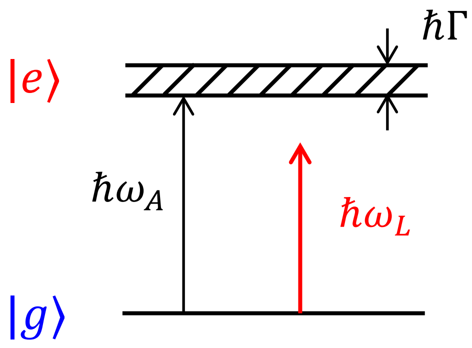

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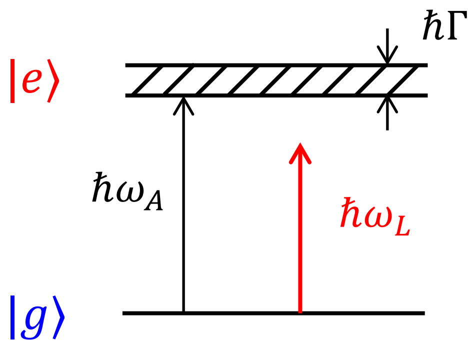

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For **most commonly used optical transitions**

$$\hbar\Gamma \gg E_{rec} \Leftrightarrow \tau_R \ll \tau_{ext}$$

Broad line condition

We can treat the internal state as stationary

Two-level model

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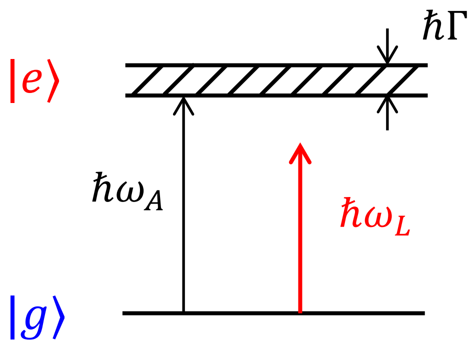

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Broad line **condition**

We can treat the internal state as stationary

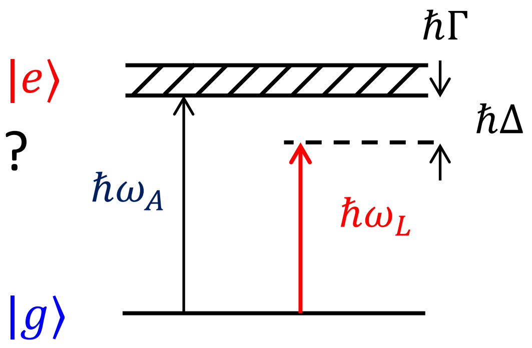
Average photon scattering rate: $\gamma = \Gamma P_e$

Average force ($\Delta\boldsymbol{p} = \hbar\boldsymbol{k}$ per scattering event): $\mathbf{F} = \Gamma P_e \hbar\boldsymbol{k}$

P_e: Stationary population in excited state

Two-level model – stationary state

- **Two-level model:**
  stationary internal state?



Γ = natural linewidth = $1/\tau_R$
τ_R = radiative lifetime of the excited state (~ 10^{-8} s)
Δ = laser ***detuning***

Pauli matrices

$$\begin{array}{cc} & \begin{array}{cc} |g\rangle & |e\rangle \end{array} \\ \hat{\sigma}_z = & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{array} \qquad \hat{\sigma}_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Atom – laser coupling :
Rabi hamiltonian

$$H = \frac{\hbar}{2}\Delta\hat{\sigma}_z + \frac{\hbar\Omega}{2}\hat{\sigma}_x$$

Electric dipole ← → **Electric field amplitude**

$$\Omega = \frac{d.\mathcal{E}}{\hbar}$$

Ω = Rabi angular frequency

Atom – field coupling :
Radiative decay

$$\frac{dP_e}{dt} = -\Gamma P_e$$

$$\frac{dP_g}{dt} = \Gamma P_g$$

A proper quantum description requires use of the density matrix:

$$\hat{\rho} = \begin{pmatrix} \rho_{gg} & \rho_{ge} \\ \rho_{eg} & \rho_{ee} \end{pmatrix}, \hat{\rho}^\dagger = \hat{\rho}, \mathrm{Tr}(\hat{\rho}) = 1$$

Two-level model – stationary state

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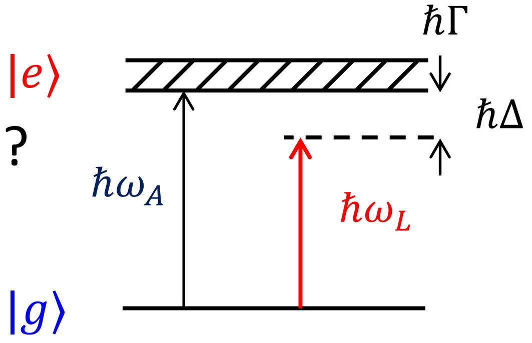

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Atom – field coupling :
Radiative decay

$$\frac{d\rho_{ee}}{dt} = -\Gamma\rho_{ee} \qquad \frac{d\rho_{ge}}{dt} = -\frac{\Gamma}{2}\rho_{ge}$$

$$\frac{d\rho_{gg}}{dt} = \Gamma\rho_{gg} \qquad \frac{d\rho_{eg}}{dt} = -\frac{\Gamma}{2}\rho_{eg}$$

$$\hat{\rho} = \begin{pmatrix} \rho_{gg} & \rho_{ge} \\ \rho_{eg} & \rho_{ee} \end{pmatrix}$$

$$\rho_{ee} = P_e$$

Combination of coherent coupling with the laser and incoherent coupling with spontaneous emission leads to the ***optical Bloch equations:***

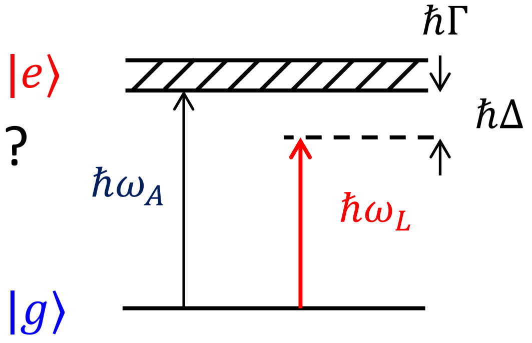
$$i\hbar\frac{d\hat{\rho}}{dt} = [\hat{\mathrm{H}}, \hat{\rho}\,] + i\hbar\left.\frac{d\hat{\rho}}{dt}\right|_{incoh}$$

$$\frac{d\rho_{ee}}{dt} = -\Gamma\rho_{ee} + \frac{i\Omega}{2}(\rho_{eg} - \rho_{ge})$$

$$\frac{d\rho_{eg}}{dt} = \left(i\Delta - \frac{\Gamma}{2}\right)\rho_{eg} - \frac{i\Omega}{2}(\rho_{gg} - \rho_{ee})$$

Two-level model – stationary state

- **Two-level model:**
  stationary internal state?



$\Gamma =$ natural linewidth = $1/\tau_R$

τ_R = radiative lifetime of the excited state (~ 10^{-8} s)

Δ = laser ***detuning***

Solving for the stationary values of optical Bloch equations gives the ***populations*** and ***coherences***

$$P_{ee} = \frac{1}{2}\frac{s}{1+s} \qquad \rho_{eg} = \frac{\Omega}{2\Delta + i\Gamma}\frac{1}{1+s}$$

With s the **saturation parameter:**

$$s = \frac{2\Omega^2}{\Gamma^2 + 4\Delta^2}$$

For s small (small I or large Δ):

$$P_e \simeq \frac{s}{2} \propto I \qquad \Omega^2 \propto \mathcal{E}^2 \propto I$$

For s large

$$P_e \simeq \frac{1}{2}$$

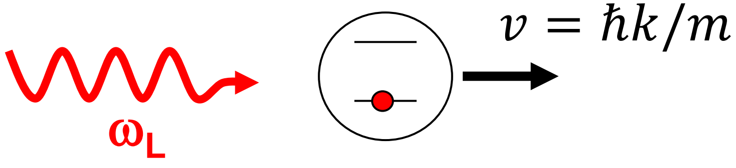
Saturation of the transition

Motion cooling

With the two-level model,

for a broad line $\hbar\Gamma \gg \frac{mv_{rec}^2}{2}$

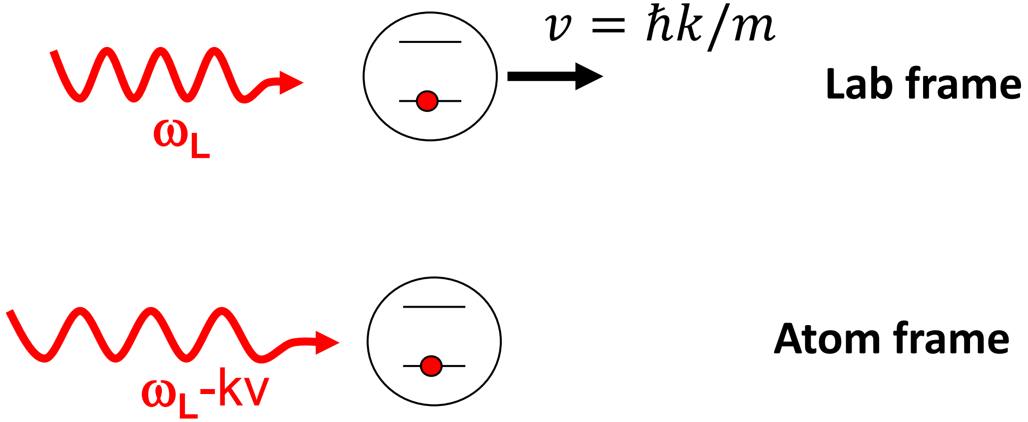
the average force is therefore



$$\mathbf{F} = \hbar k \frac{\Gamma}{2} \frac{s}{1+s}$$

$$s = \frac{2\Omega^2}{\Gamma^2 + 4\Delta^2}$$

How do we describe coupling to the motion?
Doppler effect

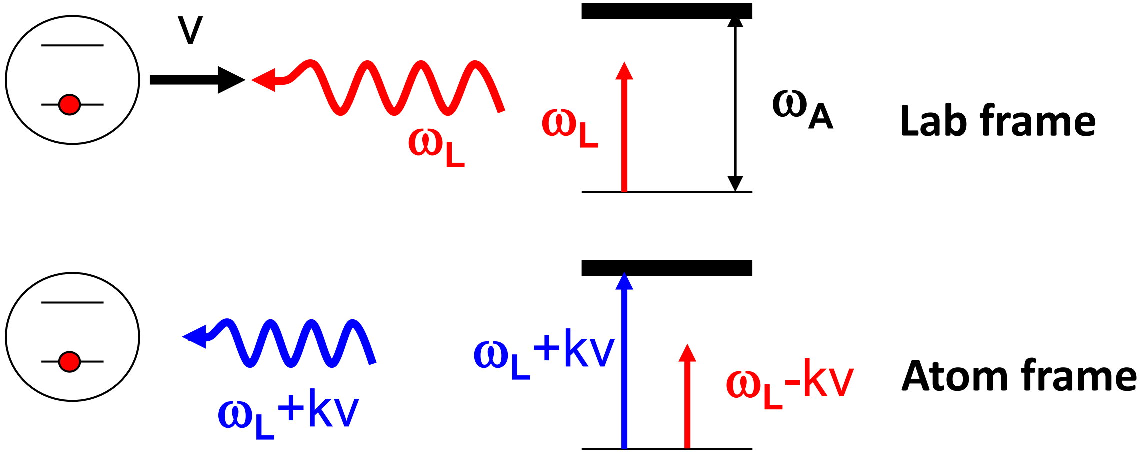


$$\Delta(\boldsymbol{v}) = \Delta - \boldsymbol{k}.\boldsymbol{v}$$

$$s(\boldsymbol{v}) = \frac{2\Omega^2}{\Gamma^2 + 4(\Delta - \boldsymbol{k}.\boldsymbol{v})^2}$$

Optical molasses

- **What laser light to choose?**

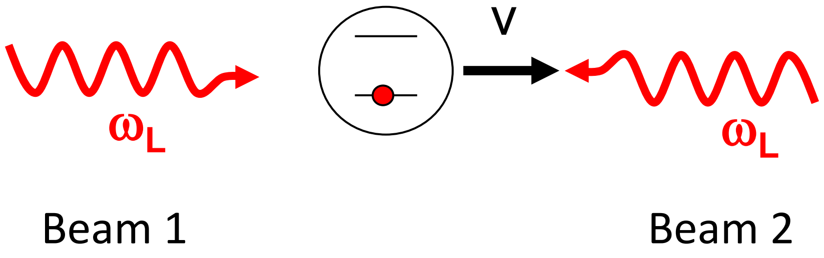


To slow down the atom:
use a ***red-detuned light*** $\boldsymbol{\omega_L < \omega_A}$

Absorption more likely if $k \cdot v < 0$
$\Longrightarrow v \cdot \Delta p < 0$

ie ***speed is reduced***

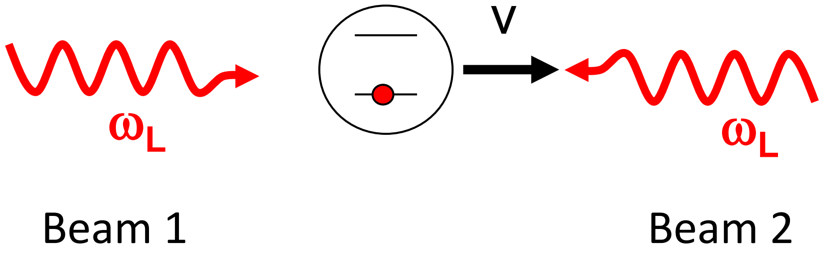
- **To reduce speed dispersion** $(\Delta v)^2 \sim \boldsymbol{T}$: counter propagating beams



In the weak saturation regime, we can add the radiative forces

$$\mathbf{F} = \boldsymbol{F_1} + \boldsymbol{F_2} \simeq \hbar \boldsymbol{k} \frac{\Gamma}{2}(\mathrm{s}_1(v) - \mathrm{s}_2(v))$$

Optical molasses



$$\mathbf{F} = \boldsymbol{F_1} + \boldsymbol{F_2} \simeq \hbar \boldsymbol{k} \frac{\Gamma}{2} (s_1(v) - s_2(v))$$

$$s(\boldsymbol{v}) = \frac{2\Omega^2}{\Gamma^2 + 4(\Delta - \boldsymbol{k}.\boldsymbol{v})^2}$$

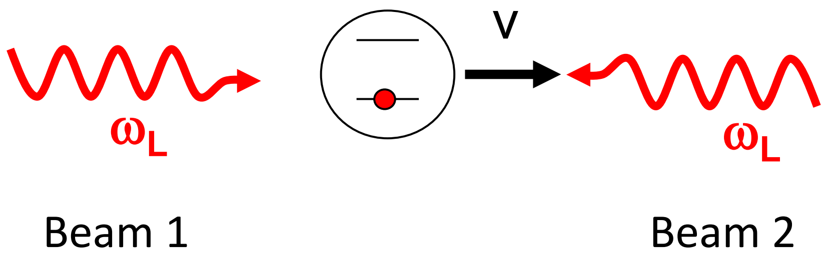
$$\mathbf{F} \simeq -m\alpha\boldsymbol{v}$$ **Viscous force**

$$m\alpha = \hbar k^2 s(0) \frac{2(-\Delta)\Gamma}{\Delta^2 + \frac{\Gamma^2}{4}}$$

We obtain a **viscous force,** which has given the name **optical molasses** to the cooled cloud. The average squared velocity is pulled to 0:

$$\frac{d(\langle v^2 \rangle)}{dt} = 2\langle v \frac{dv}{dt} \rangle = -2\alpha\langle v^2 \rangle$$

Optical molasses



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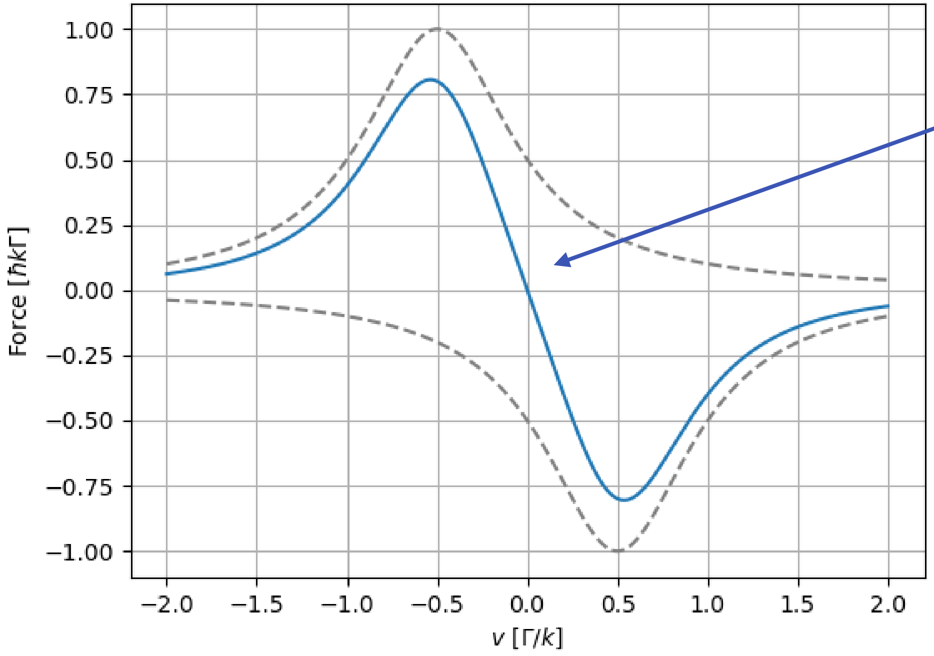


Force for $\Delta = -\frac{\Gamma}{2}$

$\Omega = \Gamma$

Central region of linear damping

However, the atoms keep absorbing and emitting: ***the final speed is not 0***

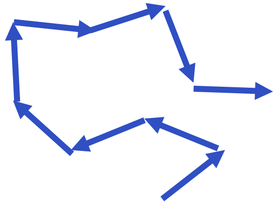
We have not accounted for the spontaneous emission photons!

Momentum diffusion

As the atom re-emits the absorbed photon:

$$\langle \Delta p \rangle \simeq 0$$

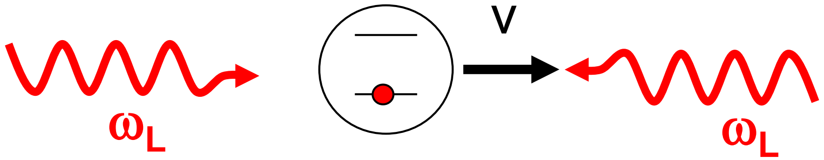
but the ***momentum still changes*** (randomly)



Random walk in momentum

Leads to **momentum diffusion:**

In 1D: for each scattering event: $\Delta\langle v^2 \rangle = \frac{\hbar^2 k^2}{m^2}$



Diffusive term for the quadratic speed (2 beams, absorption and emission):

$$\frac{d\langle v^2 \rangle}{dt} \simeq 4 \times \frac{\hbar^2 k^2}{m^2} \frac{\Gamma}{2} \mathrm{s}(0) = 2D_v$$

Doppler temperature

Combining viscous force and diffusion term, the quadratic velocity evolves as

$$\frac{d\langle v^2\rangle}{dt} = -2\alpha\langle v^2\rangle + 2D_v$$

which leads to a stationary value $\langle v^2\rangle^* = D_v/\alpha$

Equilibrium temperature $\frac{1}{2}k_B T = \frac{1}{2}m\langle v^2\rangle^*$

$$k_B T = \frac{\hbar}{2}\,\frac{\Delta^2 + \frac{\Gamma^2}{4}}{|\Delta|}$$

Minimum value ($\Delta = -\frac{\Gamma}{2}$) : $\boxed{k_B T\Big|_{Doppler} = \frac{\hbar\Gamma}{2}}$

Order of magnitude: $\Gamma \simeq 2\pi \cdot 6\text{MHz}$ (alkali)

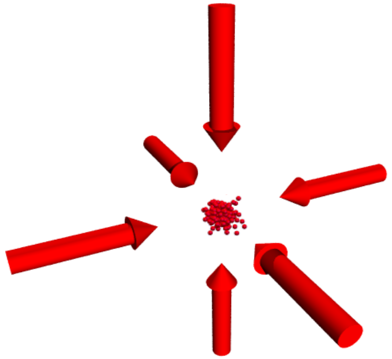
$$T\Big|_{Doppler} \simeq 150\mu\text{K}$$

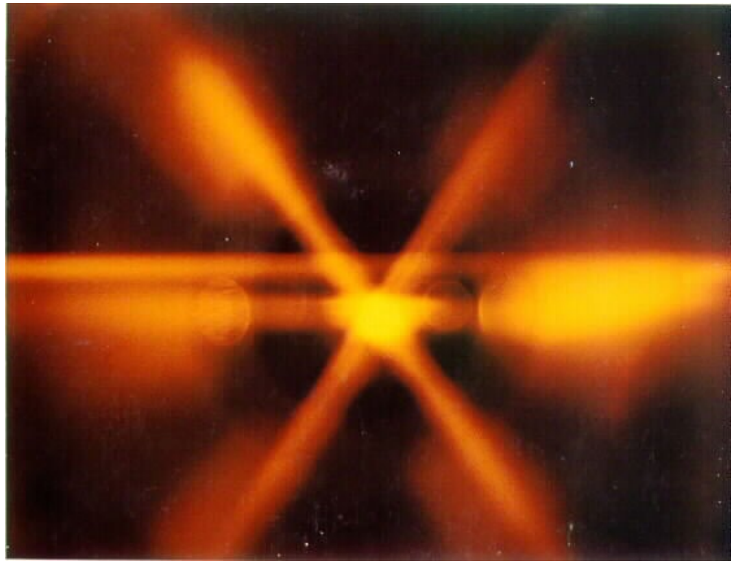
First experimental molasses

- **Cooling atoms in 3D**

Combine counter-propagating beams!

First 3D optical molasses: Bell Labs, 1985 (S. Chu *et al*)

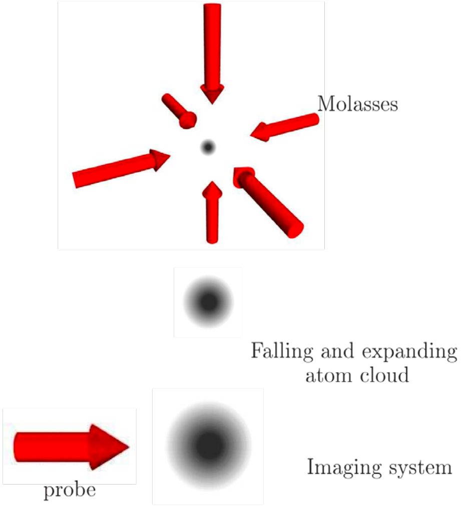


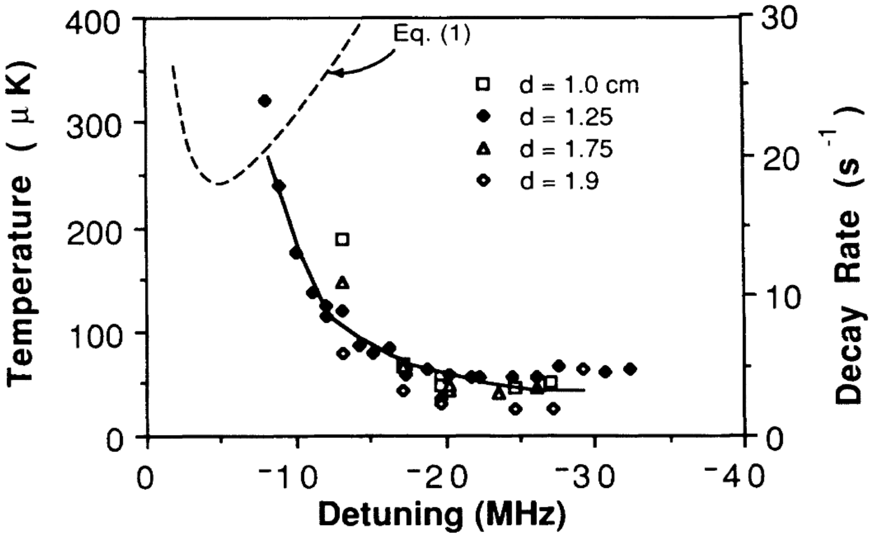


Sodium optical molasses containing a few million Na atoms at the intersection of three pairs of laser beams. (W. D. Phillips, NIST, 1987)

Some surprises…

- **First measurements** in the group of W. Phillips on sodium:



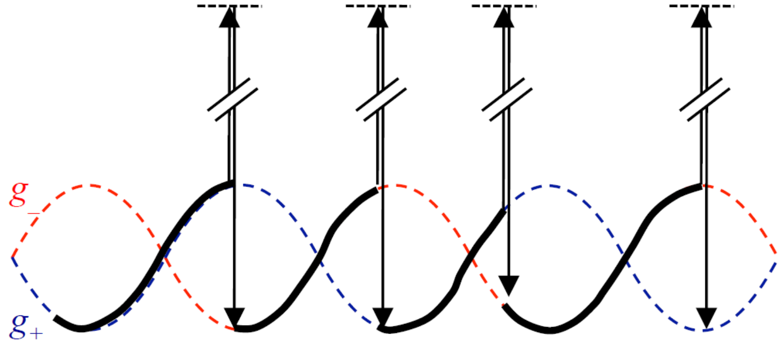


P. Lett *et al.*,
Phys.Rev. Lett. ***61****, 169*
(1988)

Measurements well below the Doppler limit!

Key elements missing:

- The ground (and excited) states have sublevels
- These sublevels experience energy shifts due to light which can vary in space (polarisation gradient)



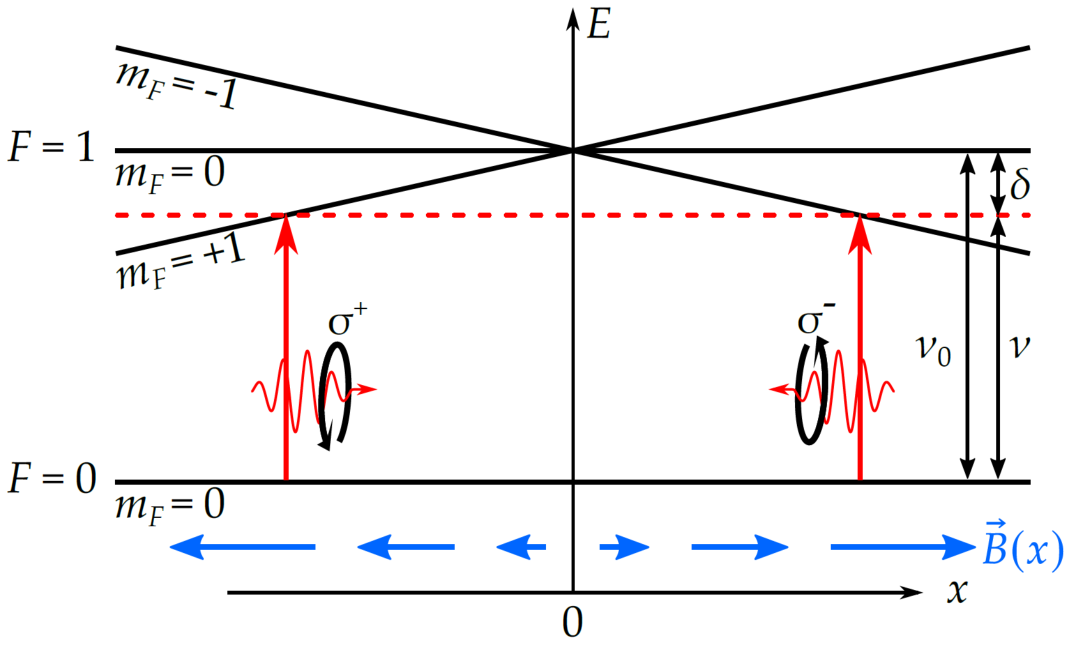
Sysiphus cooling: can decrease speeds to ~ $v_r = \hbar k/m$

Trapping in space: the magneto-optical trap

Doppler shift: speed dependent detuning → slow the atoms.
Can we have a ***space-dependent detuning***, to **trap** the atoms in one place?

Yes, by relying on magnetic sublevels, shifted by **Zeeman effect**

$$E = -\boldsymbol{\mu}.\boldsymbol{B} = m_F \times \mu_B g_F |B|$$



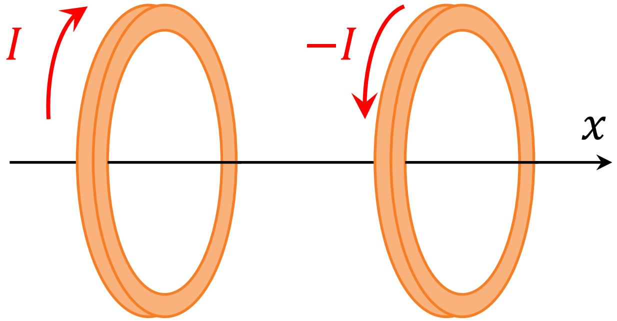
Atoms in a linear magnetic gradient: absorption more likely from counter-propagating beam as they get away from the B-field zero

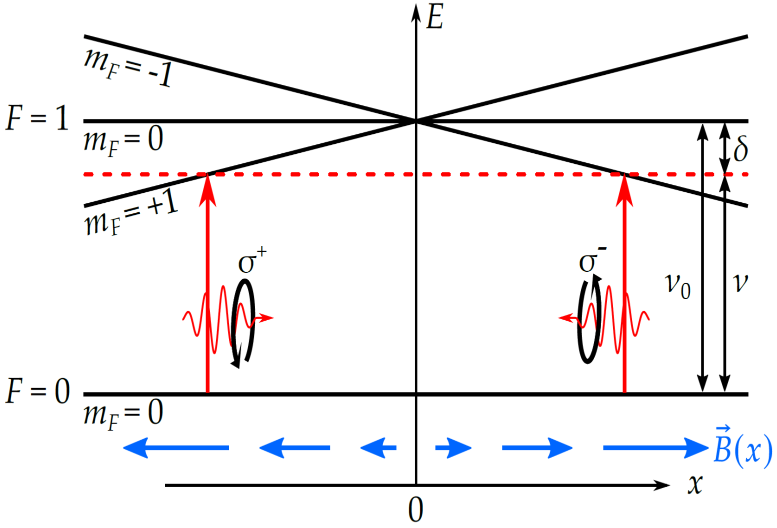
$$\boldsymbol{F = -m\alpha v - \kappa x}$$

Atoms both **cooled** and **trapped** in space

Trapping in space: the magneto-optical trap

Creation of magnetic gradient: quadrupole coils

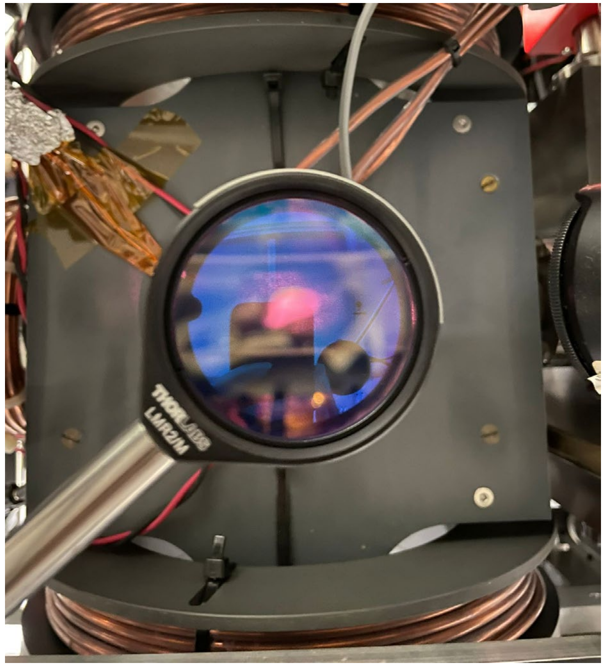




Near the center of the pair ($r \ll R$):

$$B \simeq b' \begin{pmatrix} 2x \\ -y \\ -z \end{pmatrix}$$

Atoms in the magneto-optical trap in Toulouse



Outline

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- Evaporative cooling – towards condensation
- The condensation transition
- Interacting condensates – the Gross-Pitaevskii equation

Beyond laser cooling

Laser cooling allows to reach thermal energies down to:

$$k_B T \simeq E_r = \frac{\hbar^2 k^2}{2m}$$

For rubidium, on the optical transition at 780nm:

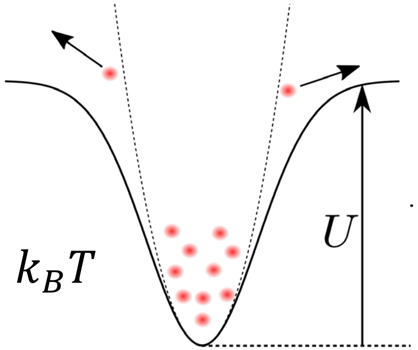
$$T \approx 1\mu\text{K}$$

This is ***not yet sufficient*** to reach condensation. We need a cooling technique that ***does not rely on absorption-emission***

Evaporative cooling

Evaporative cooling

• The concept



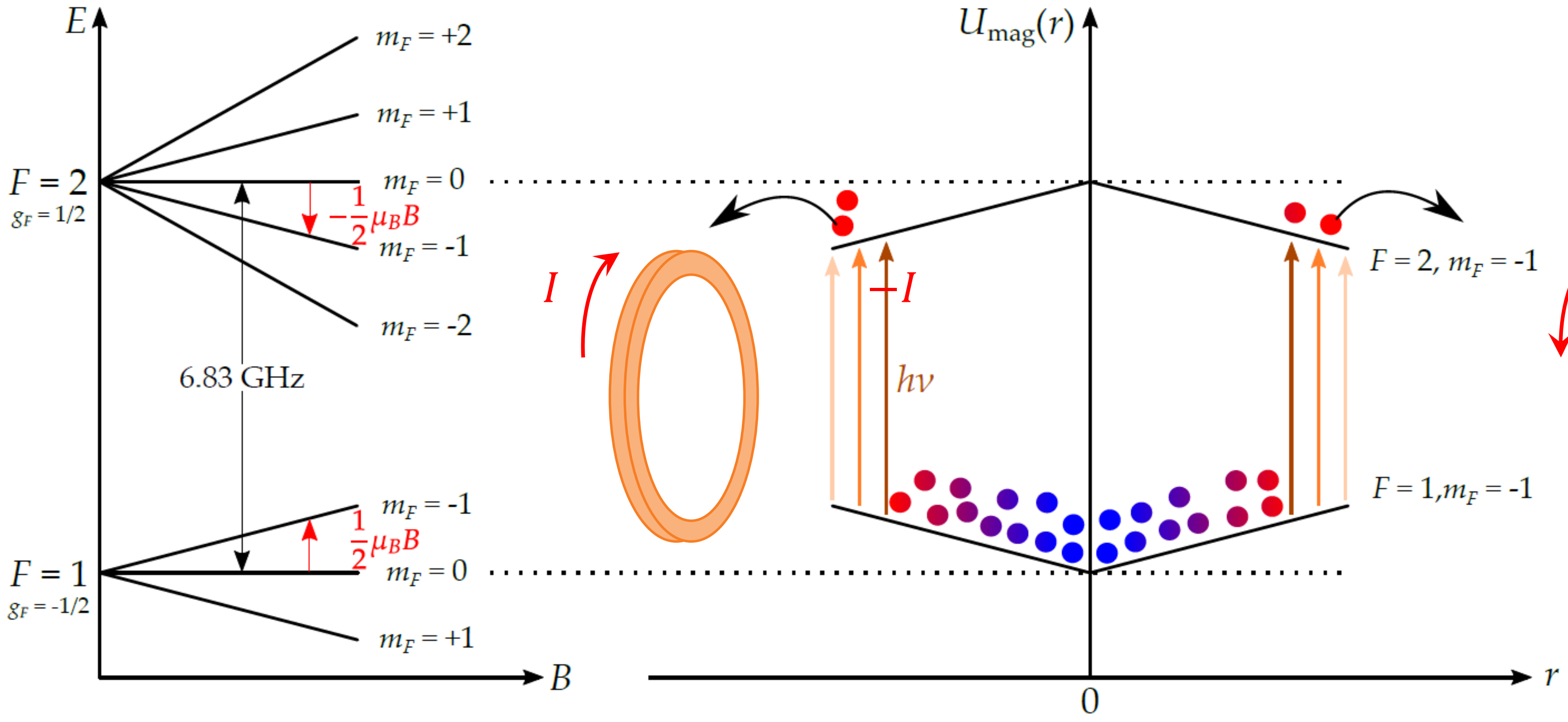
In a cold atom gas in a **trap**, if highly energetic atoms **escape** the trap the rest of the atoms will rethermalize, **through collisions**, to a lower temperature

Implementing evaporative cooling requires:

- A conservative **trap**
- A means to **remove** high-energy atoms
- Efficient **collisions** to ensure thermalization

Evaporative cooling

- **Magnetic trap**



A magnetic quadrupole can hold **low-field seeking** atoms,
and radio-frequency transitions can transfer atoms on the edge of the trap to untrapped levels.

Evaporative cooling

- **Optical trap**

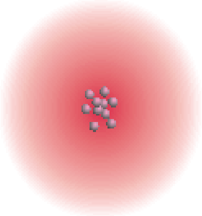
The full treatement of the atom-light interaction gives a **reactive force** related to intensity variations:

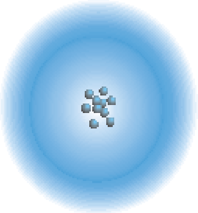
$$F = -\hbar\Delta \frac{\frac{\vec{\nabla}\Omega^2}{4}}{\Delta^2 + \frac{\Gamma^2}{4} + \frac{\Omega^2}{2}}, \qquad \Omega^2 \propto I(r)$$

This force derives from a potential:

$$F = -\vec{\nabla}U, \qquad U = \frac{\hbar\Delta}{2}\ln\left[1 + \frac{\Omega^2/2}{\Delta^2 + (\Gamma^2/4)}\right]$$

$\Delta < 0$ (red detuned)
attraction to high intensity





$\Delta > 0$ (blue detuned)
repulsion from high intensity

Dipole traps

The reactive force ***adds up to the radiation pressure*** force seen before:

For large detunings: $F_{react} \sim \frac{I(r)}{\Delta}$

$$F_{rad} \sim \frac{I(r)}{\Delta^2}$$

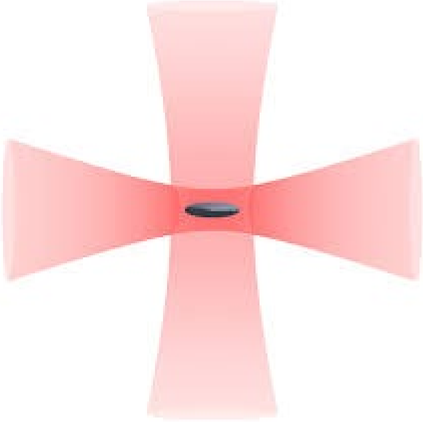
For **large detunings**, the scattering of photons is minimal, and atoms remain in their ground state and evolve in a conservative potential:

$$U \simeq \frac{\Omega^2}{4\Delta} \propto \frac{I(r)}{\Delta}$$

This force is also called the **dipole force:**

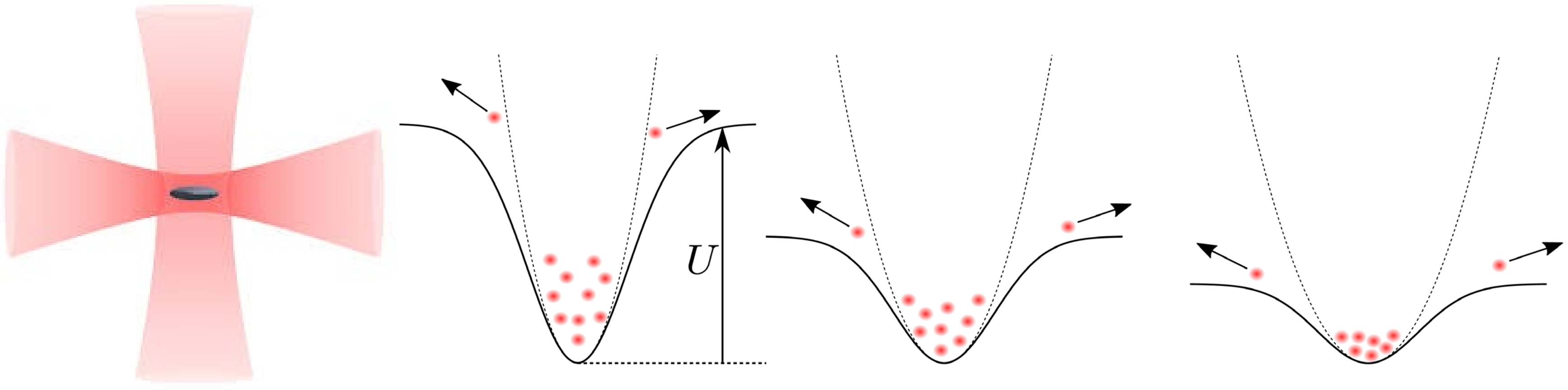
Field-induced dipole: $\boldsymbol{D} = \alpha(\omega)\boldsymbol{\mathcal{E}}$

Interaction energy: $E_{dip} = -\boldsymbol{D} \cdot \boldsymbol{\mathcal{E}} \propto I$



A crossed configuration of laser beams can trap atoms in 3D!

Evaporative cooling in a dipole trap



Reducing the beam intensity: high-energy atoms leave the trap the rest ***thermalizes to a lower temperature***

Two points of caution

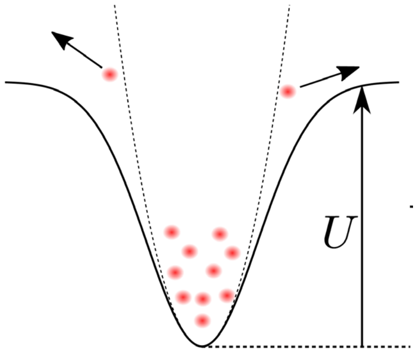
- Cooling the atoms in only interesting if $\mathcal{D} = n\lambda_T^3$ increases (to reach BEC).
  Here, as the trap opens, the density may decrease
- The timescale for evaporative cooling is set by the atom-atom collision rate:

$$\gamma_{coll} \sim n\sigma\bar{v}$$

The rate should remain high.

σ: scattering cross section of an atom ($6.5 \cdot 10^{-16}\mathrm{m}^2$ for rubidium)

Evaporative cooling : scaling laws



During evaporation, the goal is to preserve atoms at a low energy/temperature vs the effective trap depth U

$$\boldsymbol{U = \eta k_B T}$$

We consider an elementary process where high-energy atoms with energy $(\eta + \kappa)k_B T$ escape the trap, and the other atoms rethermalize

- Before: N atoms, temperature T, energy $E = \left(\frac{3}{2} + \frac{3}{2}\right) N k_B T$ (3D harmonic trap)
- After: $N - \mathrm{d}N$ atoms, energy $E - dE = 3k_B T - \mathrm{d}N(\eta + \kappa)k_B T$

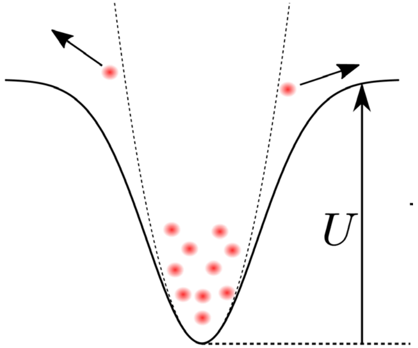
$$\frac{dE}{E} = \frac{\eta + \kappa}{3}\frac{dN}{N}$$

- After: ***new temperature*** $T - dT$

$$E - dE = 3(N - dN)k_B(T - dT)$$

$$\frac{dT}{T} = \frac{\eta + \kappa - 3}{3}\frac{dN}{N}$$

Evaporative cooling : scaling laws



If η is preserved during evaporation, we expect:

$$\frac{T_i}{T_f} = \left(\frac{N_i}{N_f}\right)^{\alpha}, \qquad \alpha = \frac{\eta+\kappa-3}{3}$$

$\kappa \sim 1$
Cooling for $\eta > 2$

Similar reasoning gives (for a 3D harmonic trap):

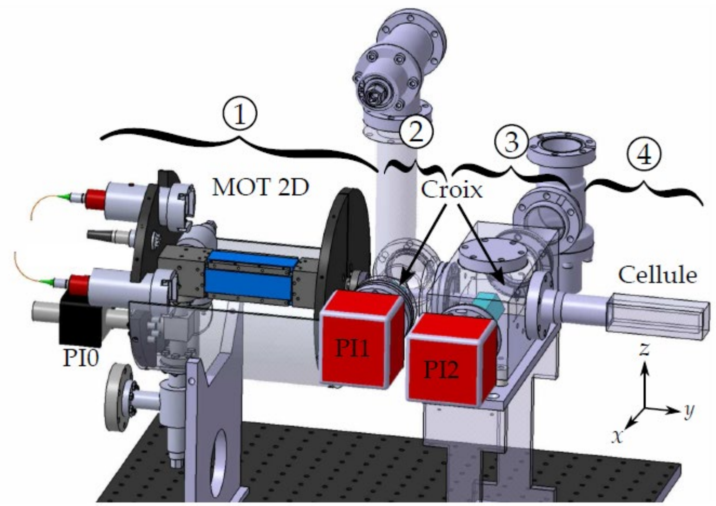
- Collision rate $\gamma_{coll} \propto \frac{N}{T} \sim N^{1-\alpha}$
- Phase space density $\mathcal{D} \propto \frac{N}{T^3} \sim N^{1-3\alpha} = N^{-\beta}$, with $\beta = \eta+\kappa-1$

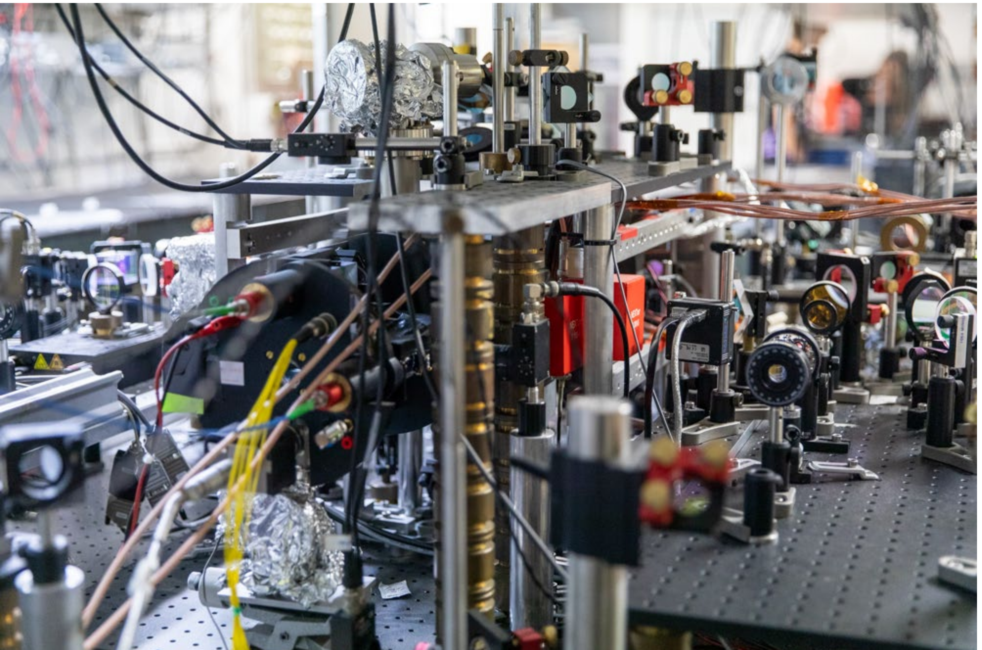
$\beta > 0$ for $\eta > 3$
phase space density increase

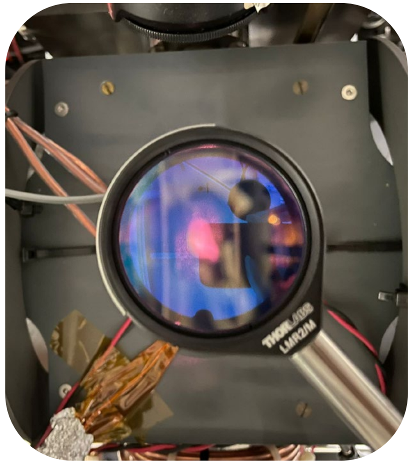
Each fractional decrease of U, $N \to N - dN$
requires a ***few collisions*** (100ms) to rethermalise:
Evaporation takes ***several seconds***

Typical numbers: $N: 10^9 \to 10^6$
$T: 100\mu K \ \to 100nK$
$\mathcal{D} \times 10^6$

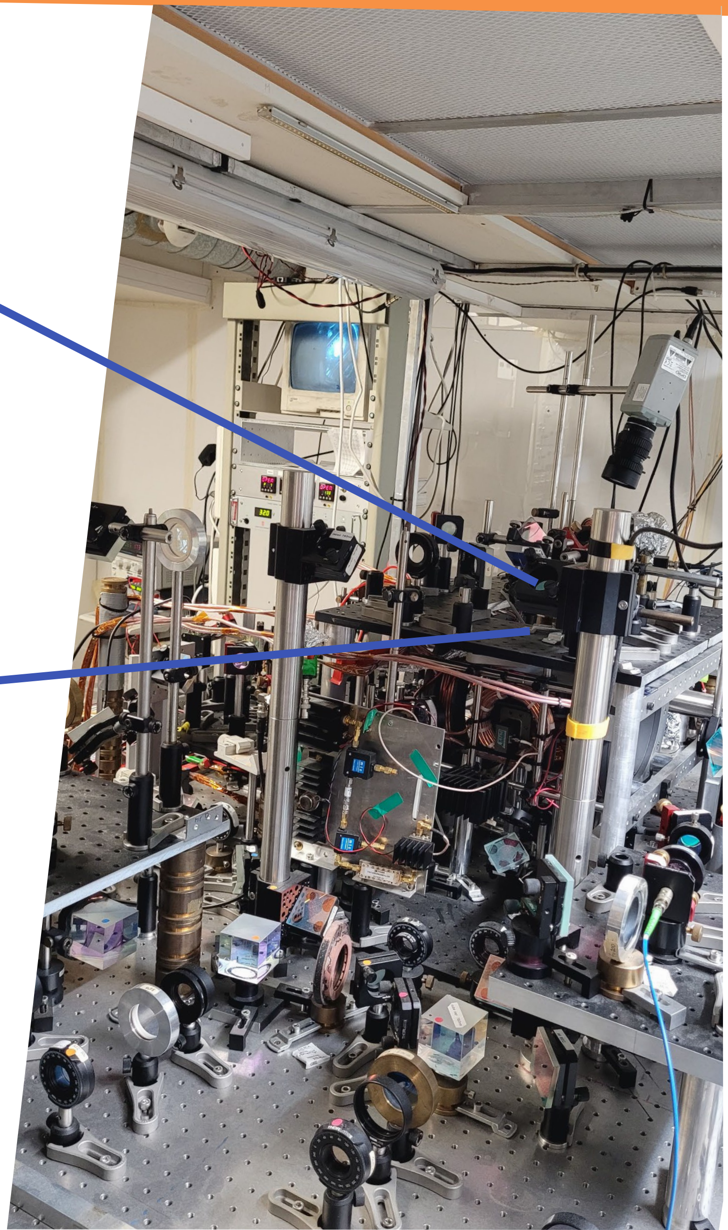
Glimpse of reality







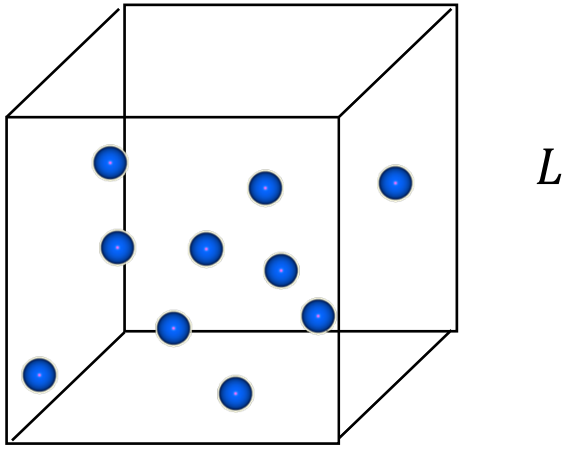
$^{87}\mathrm{Rb}$ atoms in magneto-optical trap (300 μK)



Outline

- Basics of laser (Doppler) cooling
- Evaporative cooling – towards condensation
- The condensation transition
- Interacting condensates – the Gross-Pitaevskii equation

Bosons in a 3D box



Grand canonical ensemble: each energy level ϵ is occupied with an average number given by the **Bose-Einstein distribution**

$$n_{BE}(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)}-1}$$

Physical occupation numbers → $\mu < \epsilon_0$ lowest energy level
We take $\epsilon_0 = 0$. Define $z = e^{\beta\mu}$, $0 < z < 1$, ***fugacity***

The ***total number of atoms*** writes:

$$N = N_0 + \left.\int_{0^+}\rho(\epsilon)n_{BE}(\epsilon)d\epsilon\right\} N_{exc}$$

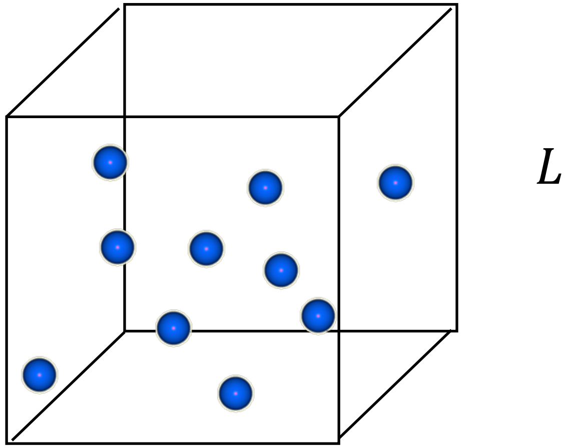
$$\rho(\epsilon) = \frac{2}{\sqrt{\pi}}\frac{V}{\lambda_T^3}\frac{\sqrt{\epsilon}}{(k_B T)^{3/2}}$$ Density of states

One can perform the integral and find:

$$N_{exc}(T) \leq N_{sat}(T) = \frac{V}{\lambda_T^3}Li_{3/2}(1) \simeq 2.612\frac{V}{\lambda_T^3}$$

Polylogarithm: $Li_n(z) = \frac{1}{n!}\int_0^\infty \frac{t^n}{\frac{e^t}{z}-1}\mathrm{dt}$

Bosons in a 3D box



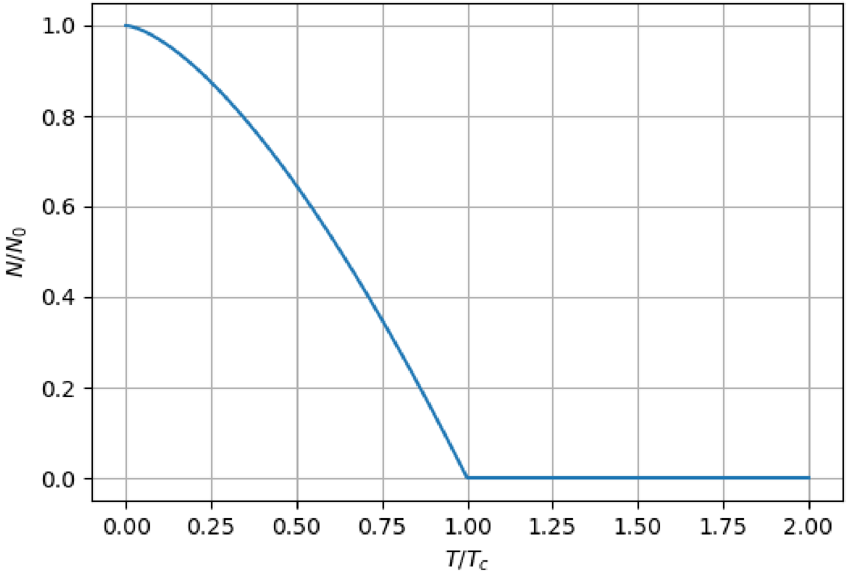
$$N_{exc}(T) \leq N_{sat}(T) \simeq 2.612\,\frac{V}{\lambda_T^3} \propto T^{3/2}$$

A ***critical temperature*** is reached when $\boldsymbol{N_{sat}(T_c) = N}$: excited levels cannot accomodate the total population

T decreases → macroscopic number of atoms in the ground state: **condensation**

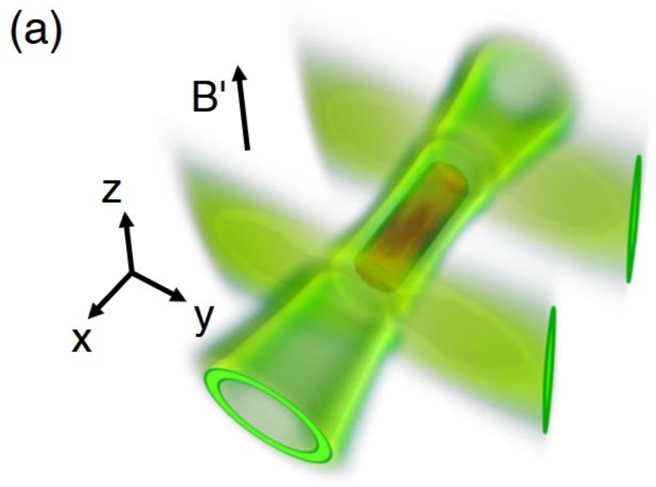
$$N = N_0 + N_{sat}(T)$$
$$= N_0 + N\left(\frac{T}{Tc}\right)^{3/2}$$

$$\Rightarrow \frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^{3/2}$$



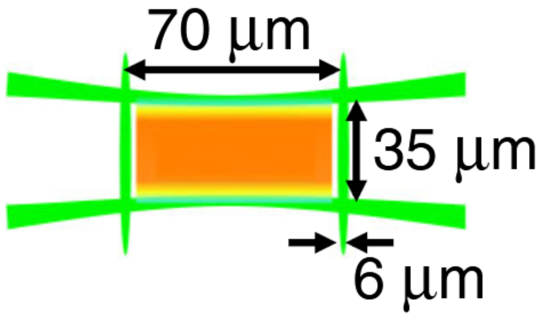
Macroscopic collective quantum state

Experiments (Cambridge)



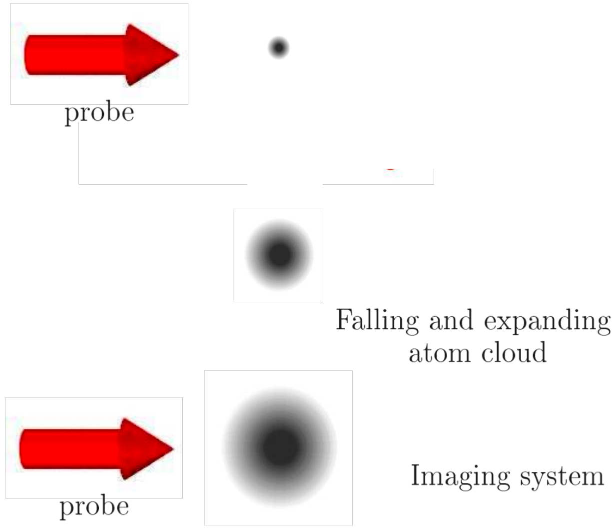
Gaunt *et al.*, *Phys. Rev. Lett.* ***110***, 200406 (2013)
(Hadzibabic group, Cambridge)

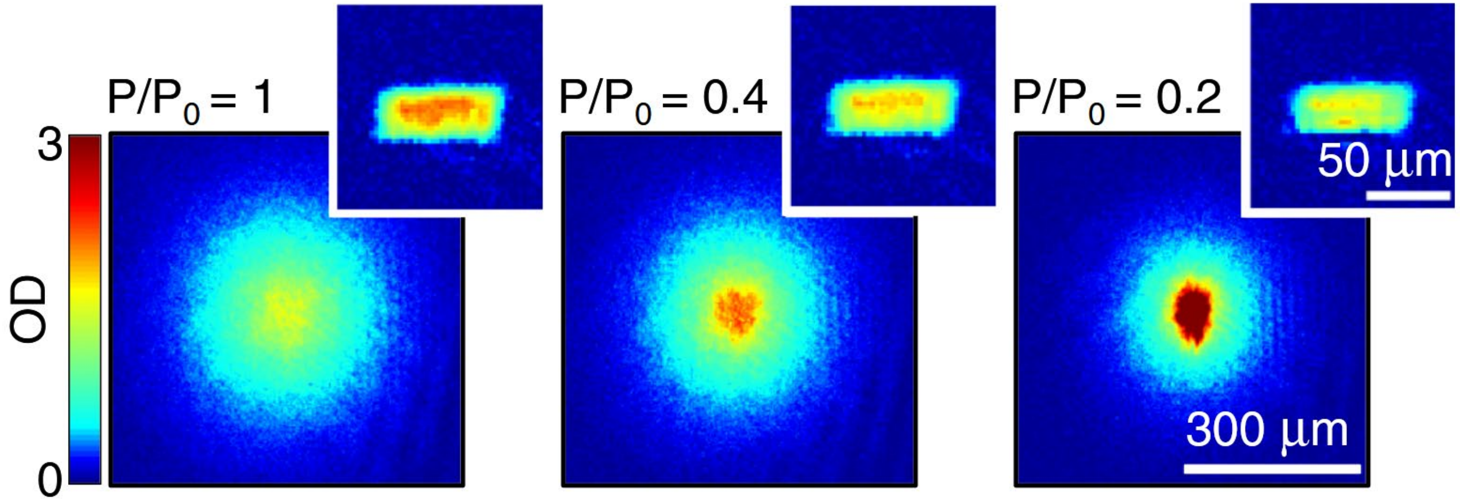
Use of a ***repulsive, green light trap*** to confine 87Rb atoms in a ‘‘**box**’’



light power P $\sim$potential depth

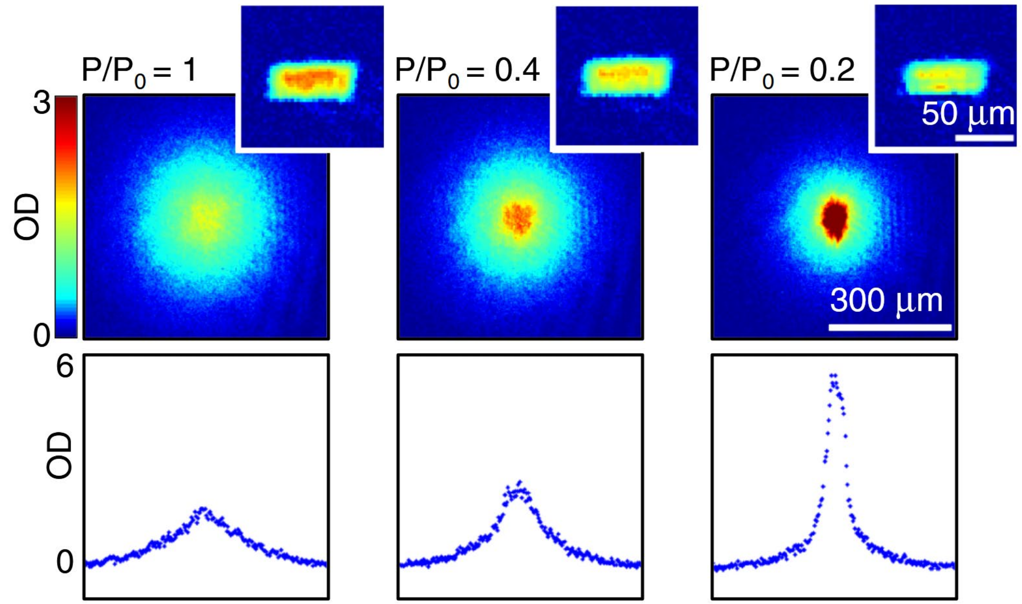
Measurement:





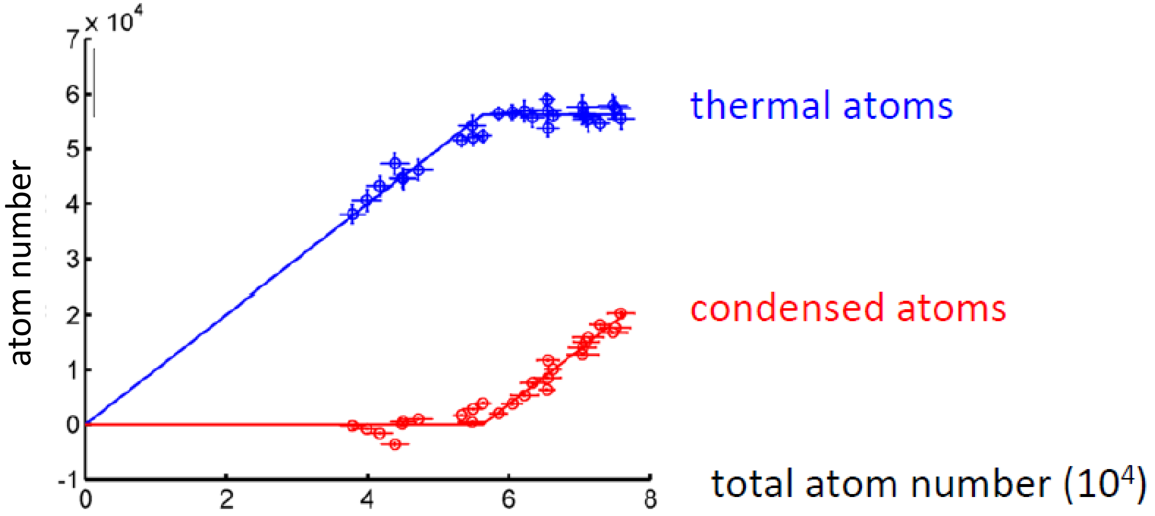
BEC appears in the momentum distribution

Experiments (Cambridge)



A ***sharp*** population near ***zero momentum*** appears, while the gas remains ***spatially homogeneous***

Repeating the experiment ***varying total number*** for the same temperature, one can ‘‘see’’ the ***saturation of excited states***



Outline

- Basics of laser (Doppler) cooling
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Atomic interactions

Atoms in the cold gas and the condensate ***interact***
(this gives the collisions for evaporation)

The true ground state (N bosons) is that of the Hamiltonian:

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + V(r_i) + \frac{1}{2} \sum_{i} \sum_{j \neq i} W(r_i - r_j)$$

Hard to solve exactly

In the condensate regime, without interaction, all atoms share the same state $|\Psi\rangle = |\psi_0(1)\rangle \otimes |\psi_0(2)\rangle \dots \otimes |\psi_0(N)\rangle$

$|\psi_0\rangle$ ground state of the trap

This is not true in general in the interacting case, but we can look for an approximation of the form

$$|\Psi\rangle = |\varphi(1)\rangle \otimes |\varphi(2)\rangle \dots \otimes |\varphi(N)\rangle$$

Hartree approximation

Energy minimization

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + V(r_i) + \frac{1}{2}\sum_{i}\sum_{j\neq i} W(r_i - r_j)$$

We search for an approximate ground state $|\Psi\rangle = |\varphi(1)\rangle \otimes |\varphi(2)\rangle \ldots \otimes |\varphi(N)\rangle$ where $|\varphi\rangle$ is a one-atom function to be determined to **minimize energy**

- Since $|\Psi\rangle$ is a product state, it **cannot describe correlations.**
  It will describe the ***mean-field*** impact on an atom from the $\mathrm{N} - 1$ others

Variational approach:

Minimize energy functional $\mathrm{E}(\varphi) = \langle\Psi|H|\Psi\rangle$, while keeping $\langle\Psi|\Psi\rangle = \langle\varphi|\varphi\rangle = 1$

Use a Lagrange multiplier and minimize: $E(\varphi) - \mu\langle\Psi|\Psi\rangle$

Energy minimization

$$E(\varphi) = E_{1body}(\varphi) + E_{interaction}(\varphi)$$

$$E_{1body}(\varphi) = N\int d^3\boldsymbol{r}\left[-\frac{\hbar^2}{2m}\varphi^*(\boldsymbol{r})\Delta\varphi(r) + V(r)|\varphi(\boldsymbol{r})|^2\right]$$

$$E_{interaction}(\varphi) = \frac{N(N-1)}{2}\iint d^3\boldsymbol{r}d^3\boldsymbol{r}'\ \varphi^*(r)\varphi^*(r')W(r-r')\varphi(r)\varphi(r')$$

Variation of $E(\varphi) - \mu\langle\Psi|\Psi\rangle$ with respect to φ and φ^* gives:

$$N\int d^3\boldsymbol{r}\,\delta\varphi^*(\boldsymbol{r})\left\{\left[-\frac{\hbar^2}{2m}\Delta\varphi(r) + V(r)|\varphi(\boldsymbol{r})|^2\right] + (N-1)\left(\int d^3\boldsymbol{r}'W(\boldsymbol{r}-\boldsymbol{r}')|\varphi(r')|^2\right)\varphi(r) - \mu\varphi(r)\right\}$$
$$+ h.c = 0$$

Variational approach

$$N\int d^3\boldsymbol{r}\,\delta\varphi^*(\boldsymbol{r})\left\{\left[-\frac{\hbar^2}{2m}\Delta\varphi(r)+V(r)|\varphi(\boldsymbol{r})|^2\right]+(N-1)\left(\int d^3\boldsymbol{r}'W(\boldsymbol{r}-\boldsymbol{r}')|\varphi(r')|^2\right)\varphi(r)-\mu\varphi(r)\right\}$$
$$+\,h.c=0$$

For a minimum, the result must hold for small variations $\delta\varphi, \delta\varphi^*$:

$$\left[-\frac{\hbar^2}{2m}\Delta\varphi(r)+V(r)|\varphi(\boldsymbol{r})|^2\right]+(N-1)\left(\int d^3\boldsymbol{r}'W(\boldsymbol{r}-\boldsymbol{r}')|\varphi(r')|^2\right)\varphi(r)=\mu\varphi(r)$$

Non-linear Schrödinger-like equation, with the wavefunction of one atom affected by the mean potential createdby the $(N-1)$ others.

Interaction potential

- For most cold atoms, interactions are short-range
- In the low energy/low temperature limit, interactions are well described by a contact **pseudo-potential**

$$W(r - r') = g\delta^{(3)}(r - r')$$

$g = \frac{4\pi\hbar^2}{m} a_s$, where a_s is the ***scattering length*** describing low-energy elastic collisions.

(we talked earlier of a collisions cross section $\sigma = 8\pi a_s^2$ for bosons; $a_s \simeq 5\ \mathrm{nm}$ for 87Rb.

This leads to ($N - 1 \simeq N$ for N large):

$$\boxed{\left[-\frac{\hbar^2}{2m}\Delta + V(r) + Ng|\varphi(r)|^2\right]\varphi(r) = \mu\varphi(r)}$$

Gross-Pitaevskii equation

Gross-Pitaevskii Equation

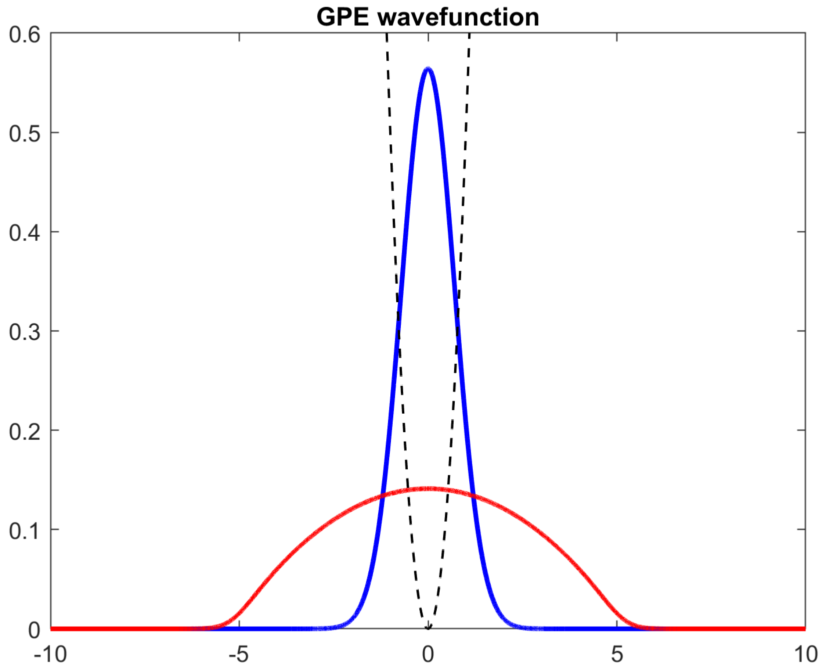
$$\left[-\frac{\hbar^2}{2m}\Delta + V(r) + Ng|\varphi(r)|^2\right]\varphi(r) = \mu\varphi(r)$$

- GP Equation is **non-linear**
- μ is the **chemical potential**:

$$\mu = \frac{dE}{dN} = \frac{\partial E[\varphi, N]}{\partial N}$$

For repulsive interactions $g > 0$, the interactions have a radical effect on the wavefunction φ:

- broadens the distribution
- ground state is not gaussian



In the limit of high interaction, φ is well-described by the **Thomas-Fermi approximation**

$$\varphi \simeq \sqrt{\frac{\mu - V(r)}{Ng}}$$

Gross-Pitaevskii equation

$$\left[-\frac{\hbar^2}{2m}\Delta + V(r) + Ng|\varphi(r)|^2\right]\varphi(r) = \mu\varphi(r)$$

- A similar approach can be taken for the time-dependent problem

$$|\Psi(\mathrm{t})\rangle = |\varphi(t,1)\rangle \otimes |\varphi(t,2)\rangle \ldots \otimes |\varphi(t,N)\rangle$$

(all atoms evolve in the same wavefunction – correlations are neglected)

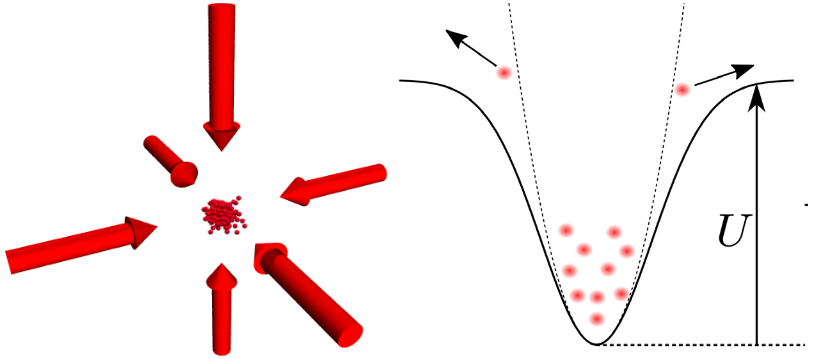
- A least-action approach gives the **time-dependent Gross-Pitaevskii equation** for the one-body wavefunction $\varphi(r,t)$:

$$\boxed{i\hbar\frac{d}{dt}\varphi(r,t) = \left[-\frac{\hbar^2}{2m}\Delta + V(r) + Ng|\varphi(r)|^2\right]\varphi(r,t)}$$

Conclusion

- **A long path to condensation**

  Cooling down 9 orders of magnitude

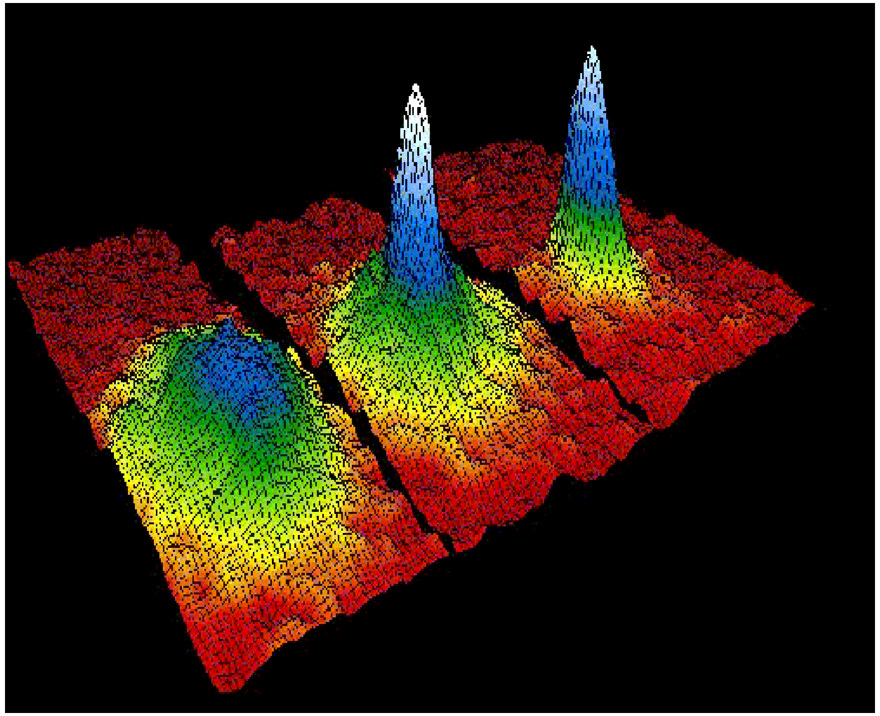


- **A phase transition from statistics**

  Saturation of excited levels

- **Importance of interactions**

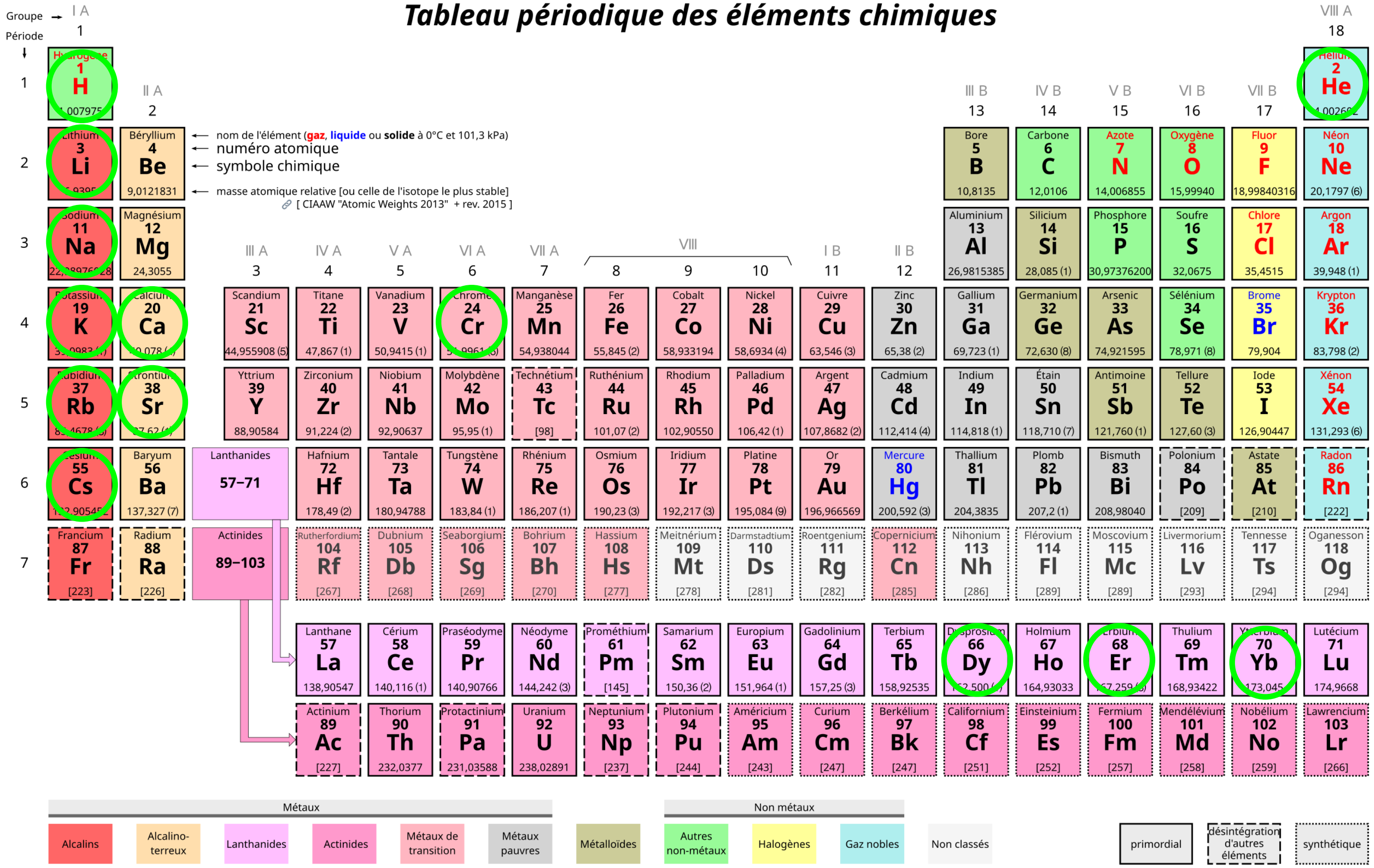
  Gross-Pitaevskii equation



The ***first BEC***, JILA (Boulder)
Science, ***269***, 198, (1995)

Conclusion

- ## Bose-Condensed elements



13 elements have been Bose-condensed

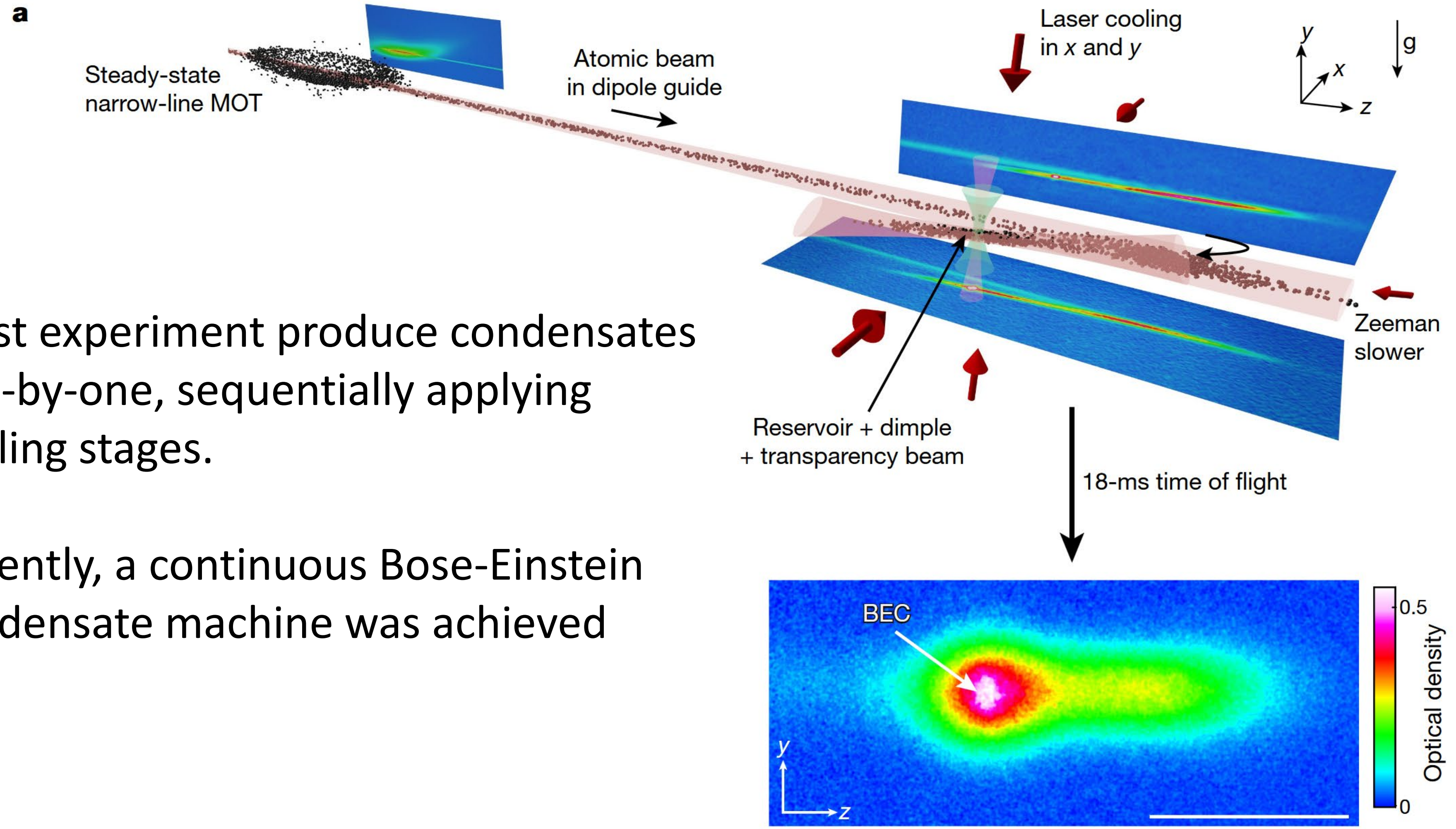
Recent interests:

- ***alkali earth*** (2 valence electrons)
- ***lanthanides*** (strong dipolar properties,

→ long range interaction)

Conclusion

- **The continuous condensate**



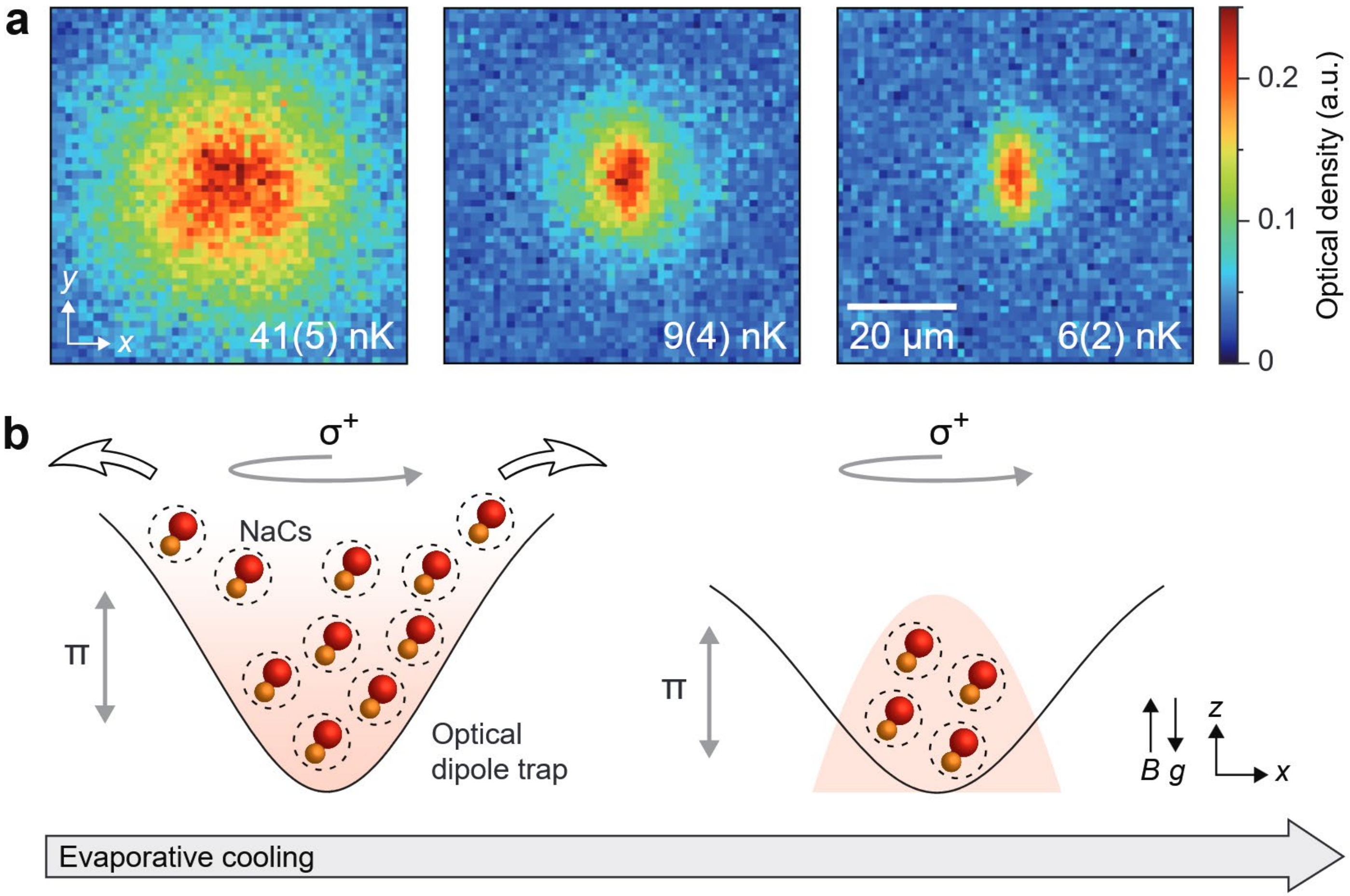
C.Chien et al,
Nature, ***606***, 683 (2022)

Most experiment produce condensates one-by-one, sequentially applying cooling stages.

Recently, a continuous Bose-Einstein condensate machine was achieved

Conclusion

- **Condensation of molecules**



Molecules have complex level structures that allow multiple interactions, collisions, chemical reactions…
→ rapid losses

Recently, the use of tailored microwave fields to block the main loss paths

First BEC of ground state molecules NaCs

Bigagli et al, *Nature* ***631***, 289 (2024)